GROUP ACTIONS AND CURVATURE
BY KARSTEN GROVE, HERMANN KARCHER AND ERNST A. RUH
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1. Introduction. The purpose of this note is to outline a proof of the following result: Any isometric action of a compact Lie group $G$ on a 1-connected, compact Riemannian manifold $M$ whose curvature tensor $R$ is sufficiently close to the curvature tensor $R_0$ of the standard sphere $S^n$ of the same dimension is equivalent to an isometric action of $G$ on $S^n$.

We measure the proximity of $R$ and $R_0$ in terms of the eigenvalues of the curvature transformation $R: V^\Lambda V \rightarrow V^\Lambda V$, where $V = T(M)$. A Riemannian manifold $M$ is called strongly $\delta$-pinched if the eigenvalues $\lambda$ of the curvature transformation at all points of $M$ satisfy the condition $\delta < \lambda \leq 1$.

2. Statement of results. The main result is as follows:

THEOREM. There exists a $\delta_0 < 1$, such that for any 1-connected, compact, strongly $\delta$-pinched $n$-dimensional Riemannian manifold $M$, and any compact Lie group $G$ the following holds:

If $\delta > \delta_0$ and $\mu: G \times M \rightarrow M$ is an isometric action of $G$ on $M$, then

1. there exists a diffeomorphism $F: M \rightarrow S^n$;
2. there exists a homomorphism $\omega: G \rightarrow O(n+1)$ such that
3. $\omega(g) = F \circ \mu(g, \cdot) \circ F^{-1}$ for all $g \in G$.

The following two corollaries are immediate consequences.

COROLLARY 1. Any compact, strongly $\delta$-pinched Riemannian manifold $M$ with $\delta > \delta_0$ is diffeomorphic to a space of constant curvature 1.

Together with Wolf's [4] classification of manifolds with constant curvature 1, this corollary gives a classification up to diffeomorphism of compact, strongly $\delta$-pinched Riemannian manifolds with $\delta > \delta_0$. In addition, the isometry group of such a manifold is isomorphic to a subgroup of the isometry group of the corresponding manifold with constant curvature.


Key words and phrases. Almost constant map, Campbell-Hausdorff formula, center of mass, conjugate group actions, strong $\delta$-pinching.

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Corollary 2. Let $M$ be a 1-connected, compact, strongly $\delta$-pinched Riemannian manifold of dimension $2n+1$ with $\delta > \delta_0$. If $S^1$ operates freely on $M$ by isometries then the quotient $M/S^1$ is diffeomorphic to the complex projective space $\mathbb{C}P^n$.

3. Outline of proof. We prove the theorem in the following steps:

(I) Construction of a preliminary diffeomorphism $f:M \to S^n$.

(II) Construction of an “almost homomorphism” $\omega_0:G \to O(n+1)$ with the property $\omega_0(g)$ is $C^1$-close to $f \circ \mu(g, \cdot) \circ f^{-1}$ for all $g \in G$.

(III) Construction of a homomorphism $\omega:G \to O(n+1)$ close to $\omega_0$.

From (II) and (III) it follows that $\omega(g) \in O(n+1) \subseteq \text{Diff}(S^n)$ and $f \circ \mu(g, \cdot) \circ f^{-1} \in \text{Diff}(S^n)$ are $C^1$-close for all $g \in G$. The corresponding actions on $S^n$ are therefore conjugate (see Grove and Karcher [1] or Palais [2]), thus there exists $S \in \text{Diff}(S^n)$ such that $\omega(g) = S \circ (f \circ \mu(g, \cdot) \circ f^{-1}) \circ S^{-1}$, i.e., $S \circ f:M \to S^n$ is the desired diffeomorphism $F$.

The ideas in (I) and (II) are based on Ruh [3]. The main tool in (III) is the notion of center of mass for almost constant maps introduced in Grove and Karcher [1]. Since (III) might be of independent interest we state it in full generality.

Let $G$ and $H$ be compact Lie groups with bi-invariant metrics normalized so that $\text{Vol}(G) = 1$, $\|[X, Y]\| \leq \|X\| \cdot \|Y\|$ for all $X, Y \in T_e H$ and such that the injectivity radius of $\exp: T_e H \to H$ is $\geq \pi$ (this choice is always possible; $\langle X, Y \rangle = \text{trace } X \circ Y^*$ in the case $H = O(n+1)$).

Proposition. If $G$ and $H$ are as above and $\omega_0:G \to H$ is a (continuous) map which is an almost homomorphism in the sense that

$$d_H(\omega_0(g \cdot g'), \omega_0(g^{-1} \cdot g')) \leq q \leq \pi/6 \quad \forall (g, g') \in G \times G$$

then there exists a (continuous) homomorphism $\omega:G \to H$ close to $\omega_0$.

In fact $d_H(\omega(g), \omega_0(g)) \leq 1.5q \quad \forall g \in G$.

Now we give a sketch of the steps in the proof of the main result.

Step (I). As in Ruh [3] we construct a flat connection $\nabla'$ on the bundle $E = T(M) \oplus 1(M)$, where $1(M)$ is the trivial line bundle $M \times \mathbb{R}$. First we define a connection $\nabla''$ with small curvature by

$$\nabla''_X Y = \nabla_X Y - \langle X, Y \rangle e, \quad \forall X, Y \in C^\infty(TM),$$

$$\nabla''_X e = X,$$

where $\nabla$ is the Riemannian connection on $T(M)$ and $e$ is the section $e_m = (O_m, 1)$. We use $\nabla''$ to construct a cross section $u'$ of the principal bundle $P$ of orthonormal $(n+1)$-frames associated to $E$, and $\nabla'$ is the corresponding flat connection of $E$. The difference $\nabla' - \nabla'' : Y \to O(n+1)$ is small for $1 - \delta$ small. We define the preliminary diffeomorphism $f:M \to S^n$.
by \( f(m) = (e, u')_m \), where \( (e, u')_m \) denotes the coordinates of \( e_m \) in the basis \( u_m \). Since \( df(X) = (\nabla'X, u') \), \( \nabla'X = X \) and \( \| \nabla' - \nabla'' \| \) is small, \( f \) is a diffeomorphism.

**Step (II).** We extend the action \( \mu : G \times M \rightarrow M \) to an action of \( G \) on \( E \) as follows: \( g \cdot (X_m + te_m) = \mu(g, \cdot)_* X_m + te_{\mu(g,m)} \). With the trivialization \( E = M \times \mathbb{R}^{n+1} \) determined by \( u \) we obtain a map \( \Omega : M \times G \rightarrow O(n+1) \); fix an arbitrary \( m_0 \in M \) and define \( \omega_0 = \Omega(m_0, \cdot) : G \rightarrow O(n+1) \). Now we estimate the deviation of \( \omega_0 \) from a homomorphism. The main observation in this estimate is that \( \nabla'' \) is invariant under the action of \( G \) and the difference \( \| \nabla' - \nabla'' \| \) is small. Now, \( \omega_0(g) \) and \( f \circ \mu(g, \cdot) \circ f^{-1} \in \text{Diff}(\mathbb{S}^n) \) are \( C^1 \)-close for all \( g \in G \) because the maps \( \Omega(m, \cdot) : G \rightarrow O(n+1) \) are almost independent of \( m \in M \).

**Step (III).** Let \( M \) and \( N \) be compact Riemannian manifolds. There exists a \( \rho' > 0 \) such that for any continuous map \( f : M \rightarrow N \) whose image is contained in a ball \( B_{\rho'} \) of radius \( \rho' \) (\( f \) is called almost constant) there is a unique point \( \mathcal{C}(f) \in B_{\rho'} \) (the center of \( f \)) with the property

\[
\int_M \exp_{\mathcal{C}(f)}(f(m)) \, dm = 0.
\]

If \( A : N \rightarrow N \) is an isometry

(*) \quad \mathcal{C}(A \circ f) = A(\mathcal{C}(f))

and if \( k : M \rightarrow M \) is a volume preserving diffeomorphism

(**) \quad \mathcal{C}(f \circ k) = \mathcal{C}(f)

see Grove and Karcher [1].

Now let \( G, H \) and \( \omega_0 : G \rightarrow H \) be as in the proposition. From \( \omega_0 \) we construct inductively a sequence \( \{ \omega_k \} \) of almost homomorphisms as follows:

\[
\omega_{k+1}(g) = \mathcal{C}(g' \rightarrow \omega_k(g \cdot g')\omega_k(g')^{-1}) \quad \forall g \in G.
\]

We prove that \( \omega_{k+1} \) is an “improvement” of \( \omega_k \) and that the sequence converges uniformly to a homomorphism \( \omega : G \rightarrow H \). The equations (*) and (**) applied to inversion, left- and right-translations reduce the proof to estimating the center \( \mathcal{C}(\eta_1 : \eta_2) \) of the product of almost constant maps with \( \mathcal{C}(\eta_1) = \mathcal{C}(\eta_2) = e \in H \). The tools used here are the Campbell-Hausdorff formula together with the comparison theorems of Rauch and Toponogov. To conclude Step (III) we apply the above proposition to the map \( \omega_0 = \Omega(m_0, \cdot) : G \rightarrow O(n+1) \). The details, as well as an estimate for the pinching constant \( \delta_0 \), will be furnished in a subsequent paper.

**ADDED IN PROOF.** In the meantime the paper has appeared under the same title in Invent. Math. 23 (1974), 31–48. Furthermore by using a Finsler norm on the orthogonal group \( O(n) \) instead of the Riemannian...
metric the theorem has been proved under the weaker assumption of sectional curvature pinching (this will appear in Math. Ann. under the title, Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems).

**REFERENCES**


**MATHEMATISCHES INSTITUT DER UNIVERSITÄT, BONN, GERMANY**