CONICAL DISTRIBUTIONS
FOR RANK ONE SYMMETRIC SPACES

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Let $X = G/K$ be a symmetric space of noncompact type, where $G$ is a connected semisimple Lie group with finite center, and $K$ is the compact part of an Iwasawa decomposition $G = KAN$ of $G$. Let $M (M')$ be the centralizer (normalizer) of $A$ in $K$. Then the space $\Xi$ of all horocycles of $X$ can be identified with $G/MN$ or $(K/M) \times A$ [1, p. 8]. The set of all smooth functions with compact supports on $\Xi$ endowed with the customary topology is denoted by $P(\Xi)$. Its dual $P'(\Xi)$ consists of all distributions on $\Xi$. Let $W$ be the Weyl group $M'/M$ and $\mathfrak{H}^*$ be the complex dual of $\mathfrak{H}$, the Lie algebra of $A$.

DEFINITION [1, p. 65]. A distribution $\psi \in P'(\Xi)$ is said to be conical if (i) $\psi$ is $MN$-invariant, (ii) $\psi$ is an eigendistribution of every $G$-invariant differential operator on $\Xi$.

As is readily seen, this definition is parallel to that of spherical functions on $X$. On this basis S. Helgason made the conjecture that the set of all conical distributions can be parametrized by $W \times \mathfrak{H}^*$, and he also established it in various cases [1, Chapter III, §4]. Our purpose here is to complete its verification in case $X$ has rank one.

Now for each $a \in A$, there is a map $\sigma(a)$ of $\Xi$ defined by $\sigma(a)(gMN) = gaMN$. This gives rise to a corresponding action $\Psi \mapsto \Psi^\sigma(a)$ on the space $P'(\Xi)$. If $\lambda \in \mathfrak{H}^*$, let $D^\lambda = \{\psi \in P'(\Xi) | \psi^\sigma(a) = e^{-(i\lambda + \rho)\log a} \psi, \forall a \in A\}$, where $\rho$ is half the sum of all positive restricted roots, counting multiplicity, and $\log : A \to \mathfrak{H}$ is the inverse of the exponential map. The space $D^\lambda$ consists of the joint eigenspaces of the $G$-invariant differential operators on $\Xi$ [1, p. 69]. So an element $\psi \in P'(\Xi)$ is conical iff it is (i) $MN$-invariant, and (ii) belongs to some $D^\lambda$. Next we recall some constructions from [1, Chapter III, §4]. For each $s \in M'/M$, choose an $m_s \in M'$ in the


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conical distributions for symmetric spaces

The group $MNA$ induces a regular Borel measure $d
u_s$ on $\Xi_s$ which is invariant under $MN$ and $\sigma(a)$, $a \in A$. For $\xi \in \Xi_s$, let $a(\xi)$ be the unique element in $A$ such that $\xi \in MNa(\xi)\xi_s$. Let $\langle \cdot, \rangle$ be the Killing form, $\Sigma^+$ the set of all positive restricted roots, $\Sigma^- = -\Sigma^+$, and $\Sigma^+_0$, the set of all indivisible roots in $\Sigma^+$.

**Theorem 1** [1, p. 82]. If $\lambda \in \mathbb{H}^*$ satisfies $\text{Re}\langle \alpha, i\lambda \rangle > 0$ for all $\alpha$ in $\Sigma^+ \cap s^{-1}\Sigma^-$, then the linear functional

$$\Psi_{\lambda,s} : \phi \mapsto \int_{\Xi_s} \phi(\xi) \exp \left( (is\lambda + s\rho)(\log a(\xi)) \right) d\nu_s, \quad \phi \in \mathcal{D}(\Xi),$$

is a conical distribution in $\mathcal{D}'_{\lambda}$.

**Theorem 2** [1, p. 88]. Let

$$d_s(\lambda) = \prod_{\alpha \in \Sigma^+_0 \cap s^{-1}\Sigma^-_0} \Gamma\left( \frac{\langle i\lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \right), \quad \lambda \in \mathbb{H}^*.$$

Then the map $\lambda \mapsto d_s^{-1}(\lambda) \Psi_{\lambda,s}$ extends from the tube $\{ \lambda \mid \text{Re}\langle i\lambda, \alpha \rangle > 0 \text{ for all } \alpha \in \Sigma^+ \cap s^{-1}\Sigma^- \}$ to a distribution valued holomorphic function $\Psi_{\lambda,s}$ on $\mathbb{H}^*_s$. For each $\lambda$, $\Psi_{\lambda,s}$ is a conical distribution in $\mathcal{D}'_{\lambda}$.

Assume, in the sequel, that the rank of $X$ equals 1. Let $s \in \mathcal{W}$ be the nontrivial element and $e \in \mathcal{W}$ the identity. Let $\alpha$ be the element in $\Sigma^+_0$, and $m_\alpha$ the multiplicity of $\alpha$. Let $d\xi$ be the $G$-invariant measure on $\Xi$.

It is noted in [1] that if $\lambda = 0$, all the $\Psi_{\lambda,s}, s \in \mathcal{W}$, constructed in Theorem 2 are proportional. The distribution $\Psi_0$ in Theorem 3 provides a compensation for this.

**Theorem 3.** For $\phi \in \mathcal{D}(\Xi)$, let $\phi_0 \in \mathcal{D}(\Xi)$ be given by $\phi_0(kaMN) = \phi(aMN)$. Then

$$\Psi_0 : \phi \mapsto \int_{\Xi} (\phi(\xi) - \phi_0(\xi)) e^{\rho(\log a(\xi))} d\xi$$

is a conical distribution in $\mathcal{D}'_0$.

From the construction we see easily that $\Psi_{\lambda,e}$ is concentrated on $\Xi_e = AMN$. So is $\Psi_{\lambda,s}$ if $-i\lambda$ is a positive integral multiple of $\alpha$. Conversely, we have

**Theorem 4.** Assume $m_\alpha \neq 1$. If the conical distribution $\Psi \in \mathcal{D}'_{\lambda}$ is
concentrated on $\Xi e$ and not proportional to $\Psi_{\lambda,e}$, then $-i\lambda$ is a positive integral multiple of $\alpha$, and $\Psi$ is a linear combination of $\Psi_{\lambda,e}$ and $\Psi_{\lambda,s}$.

(For $G = SO_0(1, n)$, $n > 2$, cf. [1, Chapter III, Theorem 4.10].)

With some more technical lemmas and the aid of Theorem 4.9 in [1, Chapter III], we finally arrive at the following main result:

**Theorem 5.** Assume the symmetric space $G/K$ has rank 1 and dimension $> 2$. Let $\Psi \in \mathcal{D}'_\lambda$ be conical. We have

(i) if $\lambda = 0$, then $\Psi = c\Psi_0 + c'\Psi_{\lambda,e};$

(ii) if $\lambda \neq 0$, then $\Psi = c\Psi_{\lambda,s} + c'\Psi_{\lambda,e},$

where $c$ and $c'$ are constants.

In case the dimension of $G/K = 2$, there is one more base element for the conical distributions in $\mathcal{D}'_\lambda$ if $i\lambda = (\frac{1}{2} - l)\alpha$, $l$ being a positive integer. This discrepancy disappears, however, if we modify the definition of conical distributions so that $G$ is the whole (not necessarily connected) isometry group. After this modification, Theorem 5 is valid for all rank one spaces. In this sense, Helgason's conjecture is true for all rank one symmetric spaces.

**REFERENCES**


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