A COMPLETE LOCAL FACTORIAL RING OF
DIMENSION 4 WHICH IS NOT COHEN-MACULAY

BY ROBERT M. FOSSUM AND PHILLIP A. GRIFFITH

Communicated by Hyman Bass, July 15, 1974

Samuel [7] stated that he knew of no factorial noetherian ring which was not Cohen-Macaulay. Murthy [6] showed that a geometric factorial ring which is Cohen-Macaulay is Gorenstein. Subsequently, Bertin [1] constructed an example of a factorial ring which was not Cohen-Macaulay. Hochster and Roberts [5] noticed that such examples abound and were found by Serre [9]. On the other hand, Raynaud, Boutot, and Hartshorne and Ogus [3] have shown that a complete local ring which is factorial, of dimension at most 4, and with \( C \) as residue class field is Cohen-Macaulay.

This note is to announce that the completion of Bertin’s example (which is characteristic 2) is factorial. This defeats a conjecture suggested by Example 5.9 of Hochster [4] which states: If \( A \) is a complete noetherian domain, then some symbolic power of a prime ideal of height one is a maximal Cohen-Macaulay module.

Let \( k \) be a perfect field of characteristic \( p \) with \( p \neq 0 \). Let \( N \) operate on \( k^4 \) by \( N(e_i) = e_{i+1} \) for \( 1 \leq i < 4 \) and with \( N(e_4) = 0 \). Then \( I + N \) is an automorphism of \( k^4 \) of order \( p \) if \( p \geq 5 \) and of order 4 if \( p = 2 \). Let \( B = k[X_1, X_2, X_3, X_4] \), which we consider to be the symmetric algebra on \( k^4 \). Let \( G \) denote the group of automorphisms of \( B \) induced by \( I + N \). It follows from Samuel [8] that the ring of invariants \( A = B^G \) is factorial. If \( p = 2 \), then Bertin [1] has shown that \( A \) is not Cohen-Macaulay. Using a result in Serre [9], Hochster and Roberts [5] show that \( A \) is not Cohen-Macaulay if \( p \geq 5 \). Let \( S = B_m \) and let \( R = S^G \), where \( m = (X_1, X_2, X_3, X_4) \). It follows that \( R = A_n \) with \( n = m \cap A \). The different \( D(S/R) = S \), and therefore the cohomology group \( H^1(G, G_m(S)) = 0 \). Let \( \hat{S} \) denote the \( m \)-adic completion of \( B \). The first result is almost obvious.
PROPOSITION 1. The n-adic completion of \( R \) is the ring of \( G \)-invariants of \( \hat{S} \). That is \( \hat{R} = \hat{S}^G \).

This yields the following corollary.

COROLLARY 2. The ring \( \hat{R} \) is not Cohen-Macaulay.

Let \( U_n = 1 + m^n S \) and \( \hat{U}_n = 1 + m^n \hat{S} \). The \( U_n \) are subgroups of \( G_m(S) \), and the following sequences are exact as \( G \)-modules:

\[
1 \to U_1 \to G_m(S) \to k^* \to 1,
\]

\[
1 \to U_{n+1} \to U_n \to m^n/m^{n+1} \to 0
\]

(and similarly with hats everywhere). Since \( G \) is cyclic, the cohomology of \( G \) is periodic of period 2 (cf. Cartan and Eilenberg [2]). We will study the exact sequence

\[
\cdots \to H^0(G, m^n/m^{n+1}) \to H^1(G, U_{n+1}) \to H^1(G, U_n) \to H^2(G, U_{n+1}) \to H^2(G, m^n/m^{n+1}) \to \cdots
\]

and the corresponding one with hats everywhere. Note that the \( G \)-module \( m^n/m^{n+1} \) is the \( n \)th symmetric power of \( m/m^2 \) as a \( G \)-module.

PROPOSITION 3. The connecting homomorphisms \( \hat{H}^0(G, m^n/m^{n+1}) \to H^1(G, U_{n+1}) \) are zero for all \( n \). Therefore the groups \( H^1(G, U_n) \) are zero and the sequence

\[
0 \to H^1(G, \hat{U}_{n+1}) \to H^1(G, \hat{U}_n) \to \cdots
\]

\[
\to H^2(G, \hat{U}_n) \to H^2(G, m^n/m^{n+1}) \to 0
\]

is exact.

REMARK. The contragredient representation of \( G \) on the \( k \)-duals of \( m^n/m^{n+1} \) induces isomorphisms of \( k \)-vector spaces:

\[
H^1(G, m^n/m^{n+1}) = H^2(G, (m^n/m^{n+1})^\vee).
\]

To show that \( \hat{R} \) is factorial, it is sufficient, therefore, to show that the homomorphisms \( H^1(G, \hat{U}_n) \to H^1(G, m^n/m^{n+1}) \) are zero for all \( n \). In characteristic \( p = 2 \), this is accomplished by directly calculating the groups \( H^1(G, m^n/m^{n+1}) \) and then showing that the connecting homomorphisms to \( H^2(G, U_{n+1}) \) are injections. Similar arguments should suffice in characteristic \( p \geq 5 \).
PROPOSITION 4. Suppose \( \text{char } k = 2 \). If \( n \) is odd, then \( H^1(G, m^n/m^{n+1}) = 0 \). If \( n \) is even, then \( \dim_k H^1(G, m^n/m^{n+1}) = \lceil n/4 \rceil + 1 \). If \( n = 4k \) and \( x = X_1(X_1 + X_2) \) (which is just \( X_1 \cdot (I + N)^2(X_1) \)), then a basis for \( H^1(G, m^n/m^{n+1}) \) is given by the classes of \( x^{2k}, x^{2k-1}a(x), \ldots, x^ka(x)^k \). A basis for \( H^2(G, m^n/m^{n+1}) \) is given by the classes of \( (x + a(x))^{2k}, x^{2k-1}a(x) + xa(x)^{2k-1}, \ldots, x^ka(x)^k \), where \( a(x) = (I + N)x \) (and similarly for \( n = 4k + 2 \)).

The results announced here, as well as similar ones for \( \mathbb{Z}/p\mathbb{Z} \) acting on \( k[[X_0, \ldots, X_{p-1}]] \), will appear elsewhere.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA, ILLINOIS 61801