AN ALGORITHM FOR THE TOPOLOGICAL DEGREE
OF A MAPPING IN \(n\)-SPACE\(^1\)

BY FRANK STENGER

Communicated by Eugene Isaacson, October 3, 1974\(^2\)

1. Introduction. In this paper we announce a new formula for computing the topological degree \(d(F, P, \theta)\), where \(F = (f^1, \cdots, f^n)\) is a vector of real continuous functions mapping a polyhedron \(P\) in \(\mathbb{R}^n\) into \(\mathbb{R}^n\), and \(\theta\) is the zero vector in \(\mathbb{R}^n\).

Let \(A = [a_{ij}]\) be an \(n \times n\) real matrix, and let \(A_i\) denote the \(i\)th row of \(A\). We use the convenient notation \(\Delta_n(A_1, \cdots, A_n)\) for the determinant of \(A\), and \(|A_i| \equiv (a_{i1}^2 + \cdots + a_{in}^2)^{1/2}\) for the Euclidean norm of \(A_i\).

Let \(X_0, X_1, \cdots, X_q\) denote \(q + 1\) points in \(\mathbb{R}^n\), where \(q \leq n\), such that the vectors \(X_i - X_0, i = 1, 2, \cdots, q\), are linearly independent. A \(q\)-simplex with vertices at \(X_0, \cdots, X_q\) is defined by

\[
S_q(X_0, \cdots, X_q) = \left\{ X \in \mathbb{R}^n : X = \sum_{i=0}^{q} \lambda_i X_i, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1 \right\}.
\]

We denote by \([X_0 \cdots X_q]\) the oriented \(q\)-simplex, defined as in [2]. For example, if \(q = n\), then \([X_0 \cdots X_q] = [X_0 \cdots X_n]\) is said to be positively (negatively) oriented in \(\mathbb{R}^n\) if \(\Delta_{n+1}(Z_0, \cdots, Z_n) > 0 (< 0)\), where \(Z_i = (1, X_i)\).

Let \(P\) be a connected, \(n\)-dimensional closed polyhedron represented as a "sum" of \(m\) positively oriented \(n\)-simplexes in the form

\[
P = \sum_{j=1}^{m'} [X_0^{(j)} \cdots X_n^{(j)}]
\]

such that the intersection of any two of the simplexes has zero \(n\)-dimensional volume.


\(^2\)Key words and phrases. Topological degree, algorithm, nonlinear equations.

\(^1\)Work supported by U. S. Army Research Grant #DAHC-04-G-017.

\(^2\)Originally received July 2, 1974.
The boundary of \([X_0 \cdots X_n]\) is represented in terms of oriented \(n-1\)-simplexes by

\[
b[X_0 \cdots X_n] = \sum_{i=0}^{n} (-1)^i[X_0 \cdots X_{i-1}X_{i+1} \cdots X_n]
\]
(see [2]). By means of this expansion, the boundary of \(P\) may be represented in the form

\[
b(P) = \sum_{j=1}^{m} t_j [Y_1^{(j)} \cdots Y_n^{(j)}]
\]

(1.2)

where \(P\) is defined in (1.1), and \(t_j = \pm 1\). For example, if \(n = 1\),

\[
P = [X_0X_1] + [X_1X_2] + \cdots + [X_{m-1}X_m],
\]

(1.3)

\[
b(P) = [X_m] - [X_0].
\]

Let \(F\) be a vector of \(n\) real \(C^1\) functions defined on \(P\), such that \(F \neq \theta = (0, \cdots, 0)\) on \(b(P)\). We denote by \(d(F, P, \theta)\) the topological degree of \(F\) at \(\theta\) relative to \(P\). We define \(d(F, P, \theta)\) by

\[
d(F, P, \theta) = \frac{1}{2} \left\{ \frac{F(X_m)}{|F(X_m)|} - \frac{F(X_0)}{|F(X_0)|} \right\}
\]

(1.4) if \(n = 1\),

\[
d(F, P, \theta) = \frac{1}{\Omega_{n-1}} \int_{b(P)} \frac{1}{|F|^n} \Delta_n \left(F, \frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_{n-1}}\right) du_1 \cdots du_{n-1}
\]

if \(n > 1\),

where \(\Omega_{n-1}\) denotes the \(n-1\) dimensional volume of the surface of the \(n\)-sphere, and where \(F = F(X(U))\) is suitably parametrized as a function of \(U = (u^1, \cdots, u^{n-1})\) (see [1, pp. 465–467]). If \(F\) is merely real and continuous on \(P\), but not necessarily of class \(C^1\), we define \(d(F, P, \theta)\) by

\[
d(F, P, \theta) = \lim_{\nu \to \infty} d(F^{(\nu)}, P, \theta),
\]

where \(F^{(\nu)}\) is real and of class \(C^1\) on \(P\) for \(\nu = 1, 2, \cdots, \max_{X \in P}|F(X) - F^{(\nu)}(X)| \to 0\) as \(\nu \to \infty\), and \(d(F^{(\nu)}, P, \theta)\) is defined by means of (1.4).

The integral formula (1.4) is due to Kronecker [1, pp. 465–467]. Another integral for \(d(F, P, \theta)\) has been given by Heinz [3]. In the following section we shall describe another procedure for evaluating \(d(F, P, \theta)\), which depends only on the sign of the components of \(F\) at a finite number of points of \(b(P)\).
2. Formula for $d(F, P, \theta)$. If $a$ is a real number, we define $\text{sgn } a$ by $\text{sgn } a = -1, 0$ or $1$ if $a < 0, = 0$ or $> 0$ respectively. We define $\text{sgn } F$ by $\text{sgn } F = (\text{sgn } f^1, \cdots, \text{sgn } f^n)$. Let us set

$$
(2.1) \quad \delta_m(F, P, \theta) = \frac{1}{2^n n!} \sum_{j=1}^{m} t_j \Delta_n(\text{sgn } F(Y^1_j), \cdots, \text{sgn } F(Y^n_j))
$$

where the $t_j$ and $Y^i_j$ are the same as in (1.2). This formula is used to compute $d(F, P, \theta)$ by means of the following

Algorithm 2.1. (1) Let $p$ be a fixed positive integer.
(2) Set $\delta = \delta_m(F, P, \theta)$ as defined in (2.1).
(3) Revise the definition of $b(P)$ as follows: For $j = 1, 2, \cdots, m$,
   (a) locate the longest segment $Y^1_k Y^1_l \cdots Y^n_k Y^n_l$ $(k < l)$ of the oriented simplex $t_j(Y^1_j \cdots Y^n_j)$ in (1.2), and set $A = \frac{Y^1_k + Y^1_l}{2}$;
   (b) replace $t_j[Y^1_j \cdots Y^n_j]$ according to:

$$
(2.2) \quad t_{j+m}[Y^1_{j+m} \cdots Y^n_{j+m}] \leftarrow t_j[Y^1_j \cdots A \cdots Y^1_l \cdots Y^n_j],
$$

where $A = \frac{Y^1_k + Y^1_l}{2}$;
(4) replace $m$ by $2m$ to get a new decomposition of $b(P)$ in terms of (twice as many) oriented simplexes.
(5) Set $e = \delta_m(F, P, \theta)$ as defined in (2.1), with the new $b(P)$.
(6) If $\delta = e$ = integer, go to Step 6. Otherwise set $\delta = e$ and return to Step 3.
(7) Replace $p$ by $p - 1$. If the resulting $p$ is positive, return to Step 3. Otherwise print out $m, \delta$.

Let us now make the following

Assumption 2.2. Let $F$ be continuous and real on $P$, where $P$ is defined as in equation (1.1). Let $b(P)$ be defined as in equation (1.2), and let $F \neq 0$ on $b(P)$. If $n > 1$, for all $1 < \mu < n$, $\phi^\mu = f^{l_i}$, $j_k \neq j_l$ if $k \neq l$, and $\Phi_\mu = (\phi^1, \cdots, \phi^\mu)$, we assume that the sets $T(A^\mu) = \{X \in b(P): \Phi_\mu(X)/\Phi_\mu(X) = a^\mu \cap S_{\mu-1} \}$ and $b(P) - T(A^\mu)$ consist of a finite number of connected subsets of $b(P)$, for all vectors $a^\mu = (\pm 1, 0, \cdots, 0), (0, \pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$, and for all $\mu - 1$-simplexes $S_{\mu-1}$ on $b(P)$.

Theorem 2.3. If Assumption 2.2 is satisfied and if the integer $p$ in Algorithm 2.2 is chosen sufficiently large, then Algorithm 2.1 prints out
finite integers $m$ and $\delta$, where $\delta = d(F, P, \theta)$, and where $P$ is defined in (1.1).

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112