SOME SUBALGEBRAS OF $L^\infty(T)$ DETERMINED BY THEIR MAXIMAL IDEAL SPACES

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1. Introduction. Sarason [4], [5] has shown, by using the notions of asymptotic multiplicity and vanishing mean oscillation, that $H^\infty(T) + C$ is determined by its maximal ideal space. In this note we announce a generalization of this result to include various superalgebras of $H^\infty(T) + C$. As intermediate steps, we develop localized notions of asymptotic multiplicity and VMO.

2. Definitions and notation. (a) Let

$$G_{n,\lambda} = \{z: 1 - 1/n < |z| < 1, |\arg z - \arg \lambda| < 1/n\}$$

for $\lambda \in T$, $n = 1, 2, \ldots$. For a closed subalgebra $A$ of $L^\infty(T)$ containing $H^\infty(T)$, the Poisson integral is said to be asymptotically multiplicative on $A$ at $\lambda$ if, for $f, g \in A$, $\epsilon > 0$, there exists an $N$ such that

$$|\hat{f}(z)\hat{g}(z) - \hat{f}(z)\hat{g}(z)| < \epsilon$$

for $z \in G_{n,\lambda}$ for all $n > N$.

(b) Now let $I$ be an arc on $T$; we define $\theta_I$ and $r_I$ such that

(i) $\theta_I$ is the center of $I$, and

(ii) $r_I = 1 - \pi m(I)$.

Now we define a collection of arcs $J_{n,\lambda} = \{\text{subarcs $I$ of $T$: } r_I e^{i\theta_I} \in G_{n,\lambda}\}$, and for $f \in L^1(T)$ we define

$$M_{n,\lambda}(f) = \sup_{I \in J_{n,\lambda}} \frac{1}{m(I)} \int_I |f - f_I| dm.$$

We say that $f \in \text{VMO}_\lambda$ if $f \in \text{BMO}$ and $\lim_{n \to \infty} M_{n,\lambda}(f) = 0$. See [4] for a definition and discussion of BMO.

(c) Let $E \subseteq T$; then $L^\infty_E(T)$ will denote the set of functions in $L^\infty(T)$ which can be extended continuously on the set $E$. When $E$ is a singleton, say $E = \{\lambda\}$, $L^\infty_E(T)$ will be denoted $L^\infty_\lambda(T)$. In case $E$ is $\sigma$-compact, it is known [2] that $H^\infty(T) + L^\infty_E(T)$ is a closed algebra.
(d) We will also be concerned with the algebra $B_{\lambda}$, defined to be the closed subalgebra of $L^\infty(T)$ generated by $H^\infty(T)$, and those functions on $T$ continuous except possibly at $\lambda$ and having two-sided limits at $\lambda$.

(e) For a closed subalgebra $A$ of $L^\infty(T)$ which contains $H^\infty(T)$, we denote the maximal ideal space of $A$ by $M(A)$, and we denote by $Y_{\lambda}$ the fibre over $\lambda$ of the maximal ideal space of $H^\infty(T)$; see [3].

3. Main results.

**Theorem 1.** Let $A$ be a closed subalgebra of $L^\infty(T)$, which contains $H^\infty(T)$; then the following are equivalent:

(i) $Y_{\lambda} \subseteq M(A)$.

(ii) The Poisson integral is asymptotically multiplicative on $A$ at $\lambda$.

**Theorem 2.** Let $w \in L^\infty(T)$, $\lambda \in T$ such that

(i) $|w(e^{i\theta})| = 1$ a.e.,

(ii) $|\hat{w}(z)|$ is continuous at $\lambda$;

then $w \in VMO_{\lambda} \cap L^\infty(T)$.

The next three theorems concern algebras determined by their maximal ideal spaces.

**Theorem 3.** Let $A$ be a closed subalgebra of $L^\infty(T)$. If

(i) $H^\infty(T) + L^\infty_{\lambda}(T) \subseteq A$ and

(ii) $M(H^\infty(T) + L^\infty_{\lambda}(T)) = M(A)$,

then $H^\infty(T) + L^\infty_{\lambda}(T) = A$.

Using Theorem 3 and some results of Davie, Gamelin and Garnett [2], we show

**Theorem 4.** Let $A$ be a closed subalgebra of $L^\infty(T)$, and let $E$ be a $\sigma$-compact subset of $T$. If

(i) $H^\infty(T) + L^\infty_{E}(T) \subseteq A$ and

(ii) $M(H^\infty(T) + L^\infty_{E}(T)) = M(A)$,

then $H^\infty(T) + L^\infty_{E}(T) = A$.

**Theorem 5.** Let $A$ be a closed subalgebra of $L^\infty(T)$. If (i) $B_1 \subseteq A$ and (ii) $M(B_1) = M(A)$, then $B_1 = A$.

4. Remark. Two students of Sarason have independently demonstrated at least some of these results. Sheldon Axler [1] has proved Theorem 4 and Alice Chang has proved Theorem 5.
REFERENCES


5. ———, *Functions of vanishing mean oscillation* (preprint).

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