A COMPARATIVE STUDY OF THE ZEROS OF DIRICHLET $L$-FUNCTIONS

BY AKIO FUJII

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We give a comparative study of the zeros of Dirichlet $L$-functions. Details will appear later.

1. Let $\chi_1$ and $\chi_2$ be distinct primitive characters of the same modulus $q$, and let $L(s, \chi_i)$, for $i = 1, 2$, be the corresponding Dirichlet $L$-functions. It is quite natural to guess that $L(s, \chi_1)$ and $L(s, \chi_2)$ have no coincident zero. In other words even a single zero will determine a Dirichlet $L$-function, or more generally, a "zeta-function". To be more precise, we call $\rho$ a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\rho, \chi_1) = L(\rho, \chi_2) = 0$ with the same multiplicities. And we call $\rho$ a noncoincident zero if it is not coincident. Then we can show

**THEOREM 1.** Let $\chi_1$ and $\chi_2$ be distinct primitive characters of the same modulus. Then a positive proportion of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are noncoincident.

Next, it is quite natural to guess that the distribution of the zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are independent. To state our results, let $\gamma_n(\chi)$ be the ordinate of the $n$th zero of $L(s, \chi)$ such that $0 \leq \gamma_n(\chi) < \gamma_{n+1}(\chi)$. Further we define $\gamma_n(\chi_1) \leq \gamma_m(\chi_2)$ if $\gamma_n(\chi_1) < \gamma_m(\chi_2)$, and $\gamma_n(\chi_1) = \gamma_m(\chi_2) \leq \gamma_{n+1}(\chi_1) \leq \gamma_{m+1}(\chi_2) \leq \cdots$ if $\gamma_n(\chi_1) = \gamma_{n+1}(\chi_1) = \cdots = \gamma_m(\chi_2) = \gamma_{m+1}(\chi_2) = \cdots$. Then we get

**THEOREM 2.** Under the same hypothesis as above, for a positive proportion of $\gamma_n(\chi_1)$'s, there does not exist a $\gamma(\chi_2)$ for which $\gamma_n(\chi_1) < \gamma(\chi_2) < \gamma_{n+1}(\chi_1)$.

Further we define $\Delta_n(\chi_1, \chi_2)$ to be $n - m$ if $\gamma_m(\chi_1) \leq \gamma_n(\chi_2) \leq \gamma_{m+1}(\chi_2)$.
\( \gamma_{m+1}(x_1) \). Then we can show

**Theorem 3.** For any positive increasing function \( \Phi(n) \) which tends to \( \infty \) as \( n \) tends to \( \infty \), we have

\[
|\Delta_n(x_1, x_2)| > 2\pi (\log \log n)^{1/2}/\Phi(n)
\]

for almost all \( n \). In particular, \( \gamma_n(x_2) \) almost never satisfies \( \gamma_n(x_1) \leq \gamma_{n+1}(x_1) \).

Theorems 1 and 2 come from a mean value theorem about

\[
\int_0^T (S(t + h, x_1) - S(t, x_1) - (S(t + h, x_2) - S(t, x_2))^i dt,
\]

where \( S(t, x) = \pi^{-1} \arg L(\frac{1}{2} + it, \chi) \) as before (cf. [1]). Theorem 3 comes from a mean value theorem about \( \int_0^T (S(t, x_1) - S(t, x_2))^j dt \). If we use mean value theorems about

\[
\sum' \sum' (S(t + h, x_1) - S(t, x_1) - (S(t + h, x_2) - S(t, x_2))^j
\]

and

\[
\sum' \sum' (S(t, x_1) - S(t, x_2))^j,
\]

where in the summation \( \chi_i \) runs over all nonprincipal characters of modulus \( q \) for each \( i = 1, 2 \), then we get \( q \)-analogues of our theorems.

2. As an application of our methods we can get some results about a problem of Knapowski-Turán. Let \( q \) be a given fixed positive integer. Assume that \( (b, q) = (d, q) = 1 \) and \( b \neq d \) (mod \( q \)). Let \( \chi \) be a character of modulus \( q \). We write \( g(\chi) = (\overline{\chi(b)} - \overline{\chi(d)})/\varphi(q) \), and \( \mu(\rho) = \mu_{b,d}(\rho) = \sum_x g(\chi)m_x(\rho) \), where \( \chi \) runs over all characters of modulus \( q \) and \( m_x(\rho) \) is the multiplicity of \( \rho \) as a zero of the Dirichlet L-functions \( L(s, \chi) \).

Knapowski and Turán proposed the following problem in their study of prime numbers:

Estimate \( f(T) = \sum_{0 < \text{Im} \rho < T, \mu(\rho) \neq 0} 1 \) (cf. [3]). Concerning this problem, Kátai (unpublished) and Grosswald [2] proved independently the existence of infinitely many \( \rho \)'s with \( \mu(\rho) \neq 0 \). Later Turán obtained the following results (cf. [6]).

1. For \( T > \psi(q) \) we have the inequality \( f(T) > c_1 \exp((\log T)^{1/5}) \).
(2) Under the assumption of the generalized Riemann hypothesis we have \( f(T) > C_2 T^{1/2} \) for \( T > \psi(q) \), where the \( C_i \) are numerical constants and \( \psi(q) \) is an explicit function of \( q \). Recently Motohashi [4] obtained the following results.

1. For \( T > \psi(q) \) we have \( f(T) > T^{1/10} (\log T)^{-3} \).
2. For any sufficiently large \( T \) there exists at least one \( q \) with \( \frac{1}{2} T^{1/2} (\log T)^{51} q T^{1/2} (\log T)^{51} \) such that \( f(T) > T^{3/28} (\log T)^{-45} \).

Now we can show

THEOREM 4. For \( T > \psi(q) \) we have \( f(T) > AT \log T \), where \( \psi(q) \) is some explicit function of \( q \) and the positive constant \( A \) may depend on \( q \).

In fact, we can take \( \psi(q) = \exp(\exp(C_1 q)) \) and \( A = \exp(-C_2 q) \) with suitable positive absolute constants \( C_1 \) and \( C_2 \).

We prove this from a mean value theorem concerning

\[
\int_0^T \left| \sum_{\chi} g(\chi) S(t + h, \chi^*) - S(t, \chi^*) \right| dt,
\]

where \( \chi^* \) is the primitive character attached to \( \chi \).

REFERENCES


SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, Rikkyo University, Tokyo, Japan