ALGEBRAS OF ANALYTIC FUNCTIONS ON DEGENERATING RIEMANN SURFACES

BY RICHARD ROCHBERG

Communicated by Richard Goldberg, September 30, 1974

I. Introduction. By a Riemann surface we mean a finite bordered Riemann surface. For a Riemann surface $S$ denoted by $A(S)$ the supremum normed Banach algebra of functions continuous on $S$ and analytic on the interior of $S$. For any two Banach spaces $A$ and $B$ define $d(A, B) = \log \inf \{\|TT^{-1}\|; T$ a continuous invertible linear map of $A$ onto $B\}$. For $S_1$ and $S_2$ homeomorphic Riemann surfaces define $d(S_1, S_2) = d(A(S_1), A(S_2))$. It is known [7] that $d$ defines a metric on $R(S_1)$, the Riemann space of $S_1$, the space of conformal equivalence classes of Riemann surfaces homeomorphic to $S_1$, and that the topology induced by this metric is the same as that induced by the Teichmüller metric. The metric space $(R(S_1), d)$ is not complete. In this note we present properties of the ideal elements that are introduced in forming the completion of the metric space $(R(S_1), d)$. Proofs of these and related results will appear in a later publication.

The main result is that the new elements are connected degenerate Riemann surfaces. In fact, the results presented strongly suggest (but do not prove) that the completion of $(R(S_1), d)$ is formed by adjoining to $R(S_1)$ exactly those elements obtained by "pinching to a point" of closed noncontractible curves on surfaces in $R(S_1)$.

On an informal geometric level these results are related to results on degeneration of compact surfaces [4] and results on boundary points of Teichmüller space [1], [2], [5].

II. An example. The following example illustrates many of the phenomena described in Theorem 2. For $0 < r < 1$ let $S_r = \{z \leq |z| < 1\}$. Let $A_r = A(S_r)$. Let $S_0$ be two closed disks with their centers identified, and let $A_0$ be the algebra of continuous functions on $S_0$ which are "analytic" on the interior.
Theorem 1. The set \( \{ A_r \}_{0 < r < 1} \) is a complete metric space with respect to the metric \( d \). This space is homeomorphic (in the natural way) to [0, 1).

This example contains a great deal of negative information about the metric \( d \).

Corollary. The metric space \(( R(S_{1/2}), d)\) is not complete. Hence the metric \( d \) on \( R(S_{1/2}) \) is not equivalent to the Teichmüller metric.

Corollary. Given \( \varepsilon \) positive one can find Banach algebras \( A \) and \( B \) with maximal ideal spaces \( M(A) \) and \( M(B) \) such that \( d(A, B) < \varepsilon \), but the homology groups of \( M(A) \) and \( M(B) \) are not isomorphic.

This example is best understood by thinking of the surfaces \( S_r \) for small \( r \) as being obtained by letting the length of the generator of the homology group tend to zero. To make this precise we need a notion of the length of a homology class.

III. Lengths of cycles. The appropriate notion of length in this context is that of “harmonic length” introduced by Landau and Osserman [3].

Let \( S \) be a Riemann surface and \( \gamma \) a smooth closed curve on \( S \). The harmonic length of \( \gamma \), \( l(\gamma) \) is defined by \( l(\gamma) = \sup \{ \int \gamma \ast du; u \text{ a real harmonic function on } S, \sup_S |u| \leq 1 \} \). \( (\ast du \text{ is the harmonic conjugate of the differential } du) \) \( l(\gamma) \) is seen to depend only on the homology class of \( \gamma \).

One easily checks that if \( \gamma_r \) is a generator of the integer homology group of the surfaces \( S_r \) in the previous example then \( l(\gamma_r) \rightarrow 0 \) as \( r \rightarrow 0 \).

Let \( S_1, S_2, \cdots \) be a sequence of homeomorphic Riemann surfaces.

Let
\[
h(\{ S_n \}) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} h_\varepsilon(S_n)
\]
where \( h_\varepsilon(S_n) \) is the dimension of the span in \( H_1(S_n, R) \) of the set of \( \gamma \) in \( H_1(S_n, Z) \) with \( l(\gamma) < \varepsilon \). Thus \( h(\{ S_n \}) \) measures the number of linearly independent homology classes which are being pinched as \( n \) becomes large.

IV. The main result. By a degenerate Riemann surface we mean the object obtained from a possibly disconnected Riemann surface by making finitely many identifications of finite point sets. Let \( S_1 \) be a Riemann surface of genus \( g \) which has \( k \) boundary contours.

Theorem 2. Let \( S_1, S_2, \cdots \) be a sequence of Riemann surfaces homeomorphic to \( S_1 \) such that \( \{ S_n \} \) is a Cauchy sequence in \( (R(S_1), d) \).
There is a Banach space $A_\infty$ such that $d(A_\infty, A(S_n)) \to 0$ as $n \to \infty$. $A_\infty$ has the following additional properties. $A_\infty$ is a point separating algebra of continuous boundary value analytic functions on the connected degenerate Riemann surface $S_\infty$ ($= \text{maximal ideal space of } A_\infty$). $\partial S_\infty$, the boundary of $S_\infty$, consists of $k$ circles. The closure of the space of real parts of functions in $A_\infty$ in the space of real valued continuous functions on $\partial S_\infty$ has codimension $2g + k - 1$. The dimension of the homology group $H_1(S_\infty, \mathbb{R})$ is $2g + k - 1 - h(S_n)$. 

The proof uses results on almost isometries of function algebras [6] to construct the algebra $A_\infty$ and then uses techniques from the theory of function algebras to describe the maximal ideal space of $A_\infty$.

One consequence of this result is that if no cycles are pinched to zero then the surfaces are not degenerating.

**Corollary.** If $h(S_n) = 0$ then $S_\infty$ is a Riemann surface homeomorphic to $S_1$.

**V. Another example.** Let $S$ be a compact Riemann surface of genus 1. Let $p$ be a point of $S$. One can choose a local uniformizer $w$ at $p$ so that the surfaces $S_n = \{q \in S \mid w(q) \geq 1/n\}, n = 1, 2, \cdots$, satisfy the hypotheses of Theorem 2. In this case the algebra $A_\infty$ will be the algebra of all functions $f$ which are continuous on $\{|z| < 1\}$, analytic in $\{|z| < 1\}$, and satisfy $f'(0) = 0$.

REFERENCES


DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130