THE NUMBER OF ZEROES OF
AN ANALYTIC FUNCTION IN A CONE

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It is not possible to estimate the number of zeroes of an analytic function of several variables defined in a cone by reducing the problem to the 1-dimensional case via Crofton's formula or similar tools of Nevanlinna theory (see e.g. [4]). We propose to extend the classical result due to Pfluger and Levin [3] using a potential theory approach.

Let $S^{m-1}$ be the unit sphere in the euclidean space $\mathbb{R}^m$, $D$ an open subset of $S^{m-1}$, $\partial D$ smooth and of bounded curvature. For $0 < r < \infty$ we set $D_r = \{x: x \in D, 0 < t < r\}$. Denote by $\rho_1 = \rho_1(D)$ the positive number such that $\rho_1(\rho_1 + m - 2)$ is the first eigenvalue of the Laplace-Beltrami operator in $D$ for the Dirichlet problem. Thus we have

THEOREM. Let $u$ be a subharmonic function in $D_r$, such that $u \neq -\infty$, $u(x) \leq A + B|x|^p$ for every $x \in D_r$. If $\rho > \rho_1$, $D'$ is an arbitrary open set, $\overline{D'} \subseteq D$, then there exists a constant $C = C(D', \rho)$ such that

$$\lim_{r \to 0} r^{-\rho - m + 2} \int_{D'_r} \Delta u \leq CB.$$  

If we identify $C^n$ with $\mathbb{R}^{2n}$ and $f$ is an analytic function in $D$, then $\log|f(z)|^2$ is subharmonic and

$$\sigma_D(r) = (n - 1)! \int_{D'_r} \Delta \log|f(z)|^2$$

represents the euclidean area of the variety $\{z \in D_r: f(z) = 0\}$. For $n = 1$, it is just the number of zeroes of $f$ in $D_r$; see [2]. Therefore we obtain the following

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COROLLARY. Let \( f \) be a nonzero analytic function in \( D_\infty \) satisfying \( |f(z)| \leq A \exp(B|z|^\rho) \) for some \( \rho > \rho_1 \); then for any \( D' \) open, \( \overline{D'} \subseteq D \), we have

\[
\lim_{r \to \infty} \sigma_{D'}(r)r^{-\rho-m+2} \leq CB.
\]

The details of the proof and related results appear in [1], here we just present the bare bones of the proof of the theorem. First, we can show that

(i) the harmonic measure of \( S_r = \{rx: x \in D\} \) at a fixed point \( x_0 \), behaves like \( O(r^{-\rho_1}) \),

(ii) if \( G_r(x) \) is the Green function of \( D_r \) with pole at \( x_0 \), \( 0 < \epsilon < 1 \) fixed, then

\[
G_r(x) \geq \text{const } r^{-\rho_1-m+2}, \quad r \to \infty
\]

for \( x \in D_{\epsilon r}, |x| > r_0 \),

(iii) we can reduce the general case to the one in which \( u \leq 0 \) on \( \partial D_\infty \).

Then we apply Green's formula, assuming \( u(x_0) \neq -\infty \),

\[
\int_{D_r} G_r(x)\Delta u = -u(x_0) + \int_{\partial D_r} u(x) \frac{\partial G_r}{\partial v}(x)
\]

(\( \partial/\partial v \) derivative in the direction of the inner normal). By (i), (iii) and the assumption on \( u \) we have

\[
\int_{D_r} G_r(x)\Delta u = O(r^{\rho_1}) \quad \text{as } r \to \infty.
\]

Applying (ii), the conclusion of the Theorem follows.

REFERENCES


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