MANIFOLDS WITH PREASSIGNED CURVATURE—
A SURVEY

BY HERMAN GLUCK

In this paper I discuss two problems of Riemannian geometry in the large concerning the existence of manifolds with preassigned curvature.

The Minkowski problem and its generalization asks in Euclidean space for a closed convex hypersurface whose curvature has been given in advance. The converse to the Gauss-Bonnet theorem asks for the existence, on a two-dimensional manifold, of a Riemannian metric with prescribed Gaussian curvature. The questions have a meeting point: the search for two-spheres in three-space with given strictly positive curvature.

While the first problem goes back to the work of Minkowski [32] in 1897, the second is of more recent vintage: it was posed explicitly by Warner in the early 1960's. Both have been solved in the last few years, and in this survey I try to give an overview and some of the details.

The paper is organized into the following sections:
1. The Minkowski problem
2. The generalized Minkowski problem
3. Converse to the Gauss-Bonnet theorem for smooth manifolds
4. Converse to the Gauss-Bonnet theorem for PL manifolds
5. Realization in three-space


(1.1) CURVATURE OF CONVEX HYPERSURFACES. Let $M^n$ be a smooth closed convex hypersurface in Euclidean space $R^{n+1}$. The Gauss map $\gamma: M^n \rightarrow S^n$ associates with each point $x \in M$ the unit outward normal vector to $M$ at $x$. Given a region $A$ on $M$, the ratio

$$\frac{\text{area of } \gamma(A) \text{ on } S^n}{\text{area of } A \text{ on } M^n}$$

represents the average curvature of $M$ throughout the region $A$, and its

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 Castelli, limit, as \( A \) shrinks down to \( x \), is the curvature of \( M \) at \( x \). For \( n \geq 3 \) this is often called the Gauss-Kronecker curvature to distinguish it from other types of curvature.

If, in addition, \( M \) is strictly convex, then \( \gamma \) is a homeomorphism and its inverse is a parametrization of \( M \). Under what conditions, Minkowski wanted to know, can one preassign a strictly positive function \( K: S^n \to \mathbb{R}^1 \) and then find \( M^n \) as above so that its curvature at the point \( \gamma^{-1}(p) \) is precisely the preassigned value \( K(p) \) for all \( p \in S^n \)?

From a slightly different view, suppose one wanted to preassign the average curvature of the set \( \gamma^{-1}(B) \) for each reasonable subset \( B \subset S^n \). It would be enough to preassign the area of \( \gamma^{-1}(B) \) on \( M \), since then the average curvature would be the value of the fraction

\[
\frac{\text{area of } B \text{ on } S^n}{\text{area of } \gamma^{-1}(B) \text{ on } M^n}.
\]

Minkowski therefore recognized that preassigning an area function in the above sense had the same spirit as preassigning a curvature function.

He began by formulating the problem in the PL category.

(1.2) PL MINKOWSKI THEOREM. Let \( n_1, \ldots, n_m \) be distinct, noncoplanar unit vectors in \( \mathbb{R}^3 \), and \( F_1, \ldots, F_m \) real numbers \( > 0 \) such that \( \sum_{i=1}^m F_i n_i = 0 \). Then there exists in \( \mathbb{R}^3 \) a closed convex polyhedral surface, unique up to translation, for which the \( n_i \) and \( F_i \) are the unit outer normals and surface areas of its faces.

The condition \( \sum_{i=1}^m F_i n_i = 0 \) can be thought of as expressing the fact that any such surface, when projected on any plane, must have an image with total area (counting algebraic signs) zero. Minkowski [32] gave a nice proof of this theorem, and Bonnesen and Fenchel [9] later recognized that with inessential changes it could be generalized to the case of convex hypersurfaces in a Euclidean space of any dimension. A. D. Alexandrov reproved this result in a striking way [6], and since his methodology has far-reaching applications, I will sketch it instead.

Alexandrov began by formulating the following geometric fact.

(1.3) LEMMA. Let \( P_1 \) and \( P_2 \) be convex polygons in the plane which cannot be moved by parallel translation so that one contains the other. Then the difference of lengths of corresponding parallel sides must change sign at least four times as one circles \( P_1 \) or \( P_2 \).

The convention in effect here is that any two convex polygons may be regarded as having pairwise parallel sides, some of which may have length 0. Thus, comparing a diamond with a square, each has eight sides, four of which are of length 0. The two figures alternate as to which has
the larger corresponding side, so that the sign mentioned in the statement of the lemma changes eight times.

Next Alexandrov restated the following result of Cauchy.

(1.4) Cauchy Lemma. Let \( L \subseteq S^2 \) be a finite graph having no circular edges and having no region of \( S^2 - L \) bounded by just two edges. Mark the edges of \( L \) randomly with + or −. Let \( N_v \) denote the number of changes of sign, as one circles \( v \), of the edges touching that vertex. Let \( N = \sum_v N_v \) denote the total number of sign changes, and \( V \), the number of vertices of \( L \).

Then

\[ N \leq 4V - 8. \]

In particular, it is impossible that each \( N_v \geq 4 \).

This lemma was proved and used by Cauchy [12] in the course of showing that all convex polyhedra in \( R^3 \) are rigid, and is done by a simple combinatorial argument, which the reader can find in [6] or [31].

Next Alexandrov tackled the uniqueness part of the proof.

(1.5) Uniqueness. Two convex polyhedra in \( R^3 \), whose faces are pairwise parallel and of equal area, differ at most by a parallel translation.

To separate out the essential part of this argument, pretend that each pair of parallel faces has pairwise parallel edges. Pick one of the polyhedra, say \( P_1 \), and mark its edges + or −, according as they are longer or shorter than the corresponding edges of \( P_2 \), and leave the edge unmarked if equal. Circling around the boundary of a face of \( P_1 \), we get no marked edges if that face is congruent via parallel translation to the corresponding face of \( P_2 \). Otherwise, since they have equal area, neither can be moved by parallel translation so as to contain the other, and the previous lemma then promises at least four sign changes around the boundary.

The spherical image \( P_1^* \) of \( P_1 \) under the Gauss map is combinatorially dual to it, and one then applies the Cauchy Lemma to \( P_1^* \), first deleting the images of unmarked edges. The simple hypotheses of the Cauchy Lemma are easily seen to be satisfied. The conclusion is that no edge of \( P_1 \) was marked, and therefore that each face of \( P_1 \) is congruent via parallel translation to the corresponding face of \( P_2 \). It follows immediately that \( P_1 \) and \( P_2 \) are themselves congruent via parallel translation.

The pretense of pairwise parallel edges is avoided in a strict proof by introducing the polyhedron \( P = (P_1 + P_2)/2 \) as a bookkeeping device. Its virtue lies in the fact that \( P^* \) is the greatest common subdivision of \( P_1^* \) and \( P_2^* \). Details may be found in [6].

(1.6) Remark. Unfortunately, the above uniqueness argument does not work in higher dimensions. To see this, note that what was essentially
proved is that if two convex polyhedra in $R^3$ have pairwise parallel faces, neither properly embeddable in the other via parallel translation, then these polyhedra must in fact be congruent via parallel translation. But this is already false in four-space, as the reader may see by comparing the hypercube of side 2 with a rectangular parallelepiped of sides 1, 1, 3, 3. Nevertheless, the uniqueness theorem is true in all dimensions via a different proof, namely Minkowski's.

(1.7) Existence. It is the existence part of Alexandrov's argument that is so striking. Let $P$ be a convex polyhedron in $R^3$. The distance from the origin to the plane containing the $i$th face of $P$ is denoted by $h_i$, with a positive sign if the origin and the rest of $P$ are on the same side of this plane, and a negative sign if not. The numbers $h_1, \cdots, h_m$ are called the support numbers of $P$. If the outer normals $n_1, \cdots, n_m$ to the faces of $P$ are known, then $P$ may be reconstructed once its support numbers are known.

Return to the statement of the PL Minkowski theorem. We are told the (noncoplanar) unit outer normals $n_1, \cdots, n_m$ and the face areas $F_1, \cdots, F_m$ of a proposed convex polyhedron in $R^3$, subject to the relation $\sum_i F_i n_i = 0$, and must demonstrate the existence of such a polyhedron.

Keep $n_1, \cdots, n_m$ fixed but let $F_1, \cdots, F_m$ vary, still subject to the relation. Then we get a whole family of problems, that is, a problem space

$$\mathcal{S} = \{(F_1, \cdots, F_m):\ each\ F_i \geq 0 \ and \ \sum_i F_i n_i = 0\},$$

which can be viewed as an open convex set in some $R^{m-3}$. It is nonempty because there is certainly a convex polyhedron circumscribed about the unit sphere and having unit outer normals $n_1, \cdots, n_m$ to its faces.

By contrast, the solution space should simply be the set of all convex polyhedra in $R^3$ with these face normals. They may be parametrized by their support numbers $(h_1, \cdots, h_m)$. Conversely, given such support numbers, we may draw in the planes $n_i \cdot \tilde{x}_i = h_i$ and expect to see $P$ emerge as the intersection of the corresponding half spaces $n_i \cdot \tilde{x}_i \leq h_i$. Of course it may happen that some of the faces of $P$ will not appear, since the intersection may equal an intersection with fewer terms. In any case, it is clear that the polyhedra $P$ actually exhibiting all $m$ faces correspond to an open subset $\mathcal{S}_0$ of the parameter space $R^m = \{(h_1, \cdots, h_m)\}$. If one now reckons two polyhedra equivalent when congruent via a parallel translation, and views this on the parameter space $\mathcal{S}_0$, one gets a quotient space

$$\mathcal{S} = \mathcal{S}_0/\text{parallel translation},$$

which is easily seen to correspond to an open subset of $R^{m-3}$. $\mathcal{S}$ is the
solution space for the given family of problems, also nonempty by circum­
scription about the unit sphere in $R^3$.

What one wants to show is that each problem has a solution. What is
in any case clear, Alexandrov points out, is that each solution has a
problem. This is expressed by a map $\varphi: \mathcal{I} \to \mathcal{P}$, which is easily seen to be
continuous. The uniqueness half of the proof, already settled, indicates
that $\varphi$ is one-to-one. Since $\mathcal{I}$ and $\mathcal{P}$ are both manifolds of dimension
$m-3$, "invariance of domain" implies that $\varphi$ is an open map.

To see that $\varphi(\mathcal{I})$ is a closed set is easy. But then $\varphi(\mathcal{I})$, as a nonempty
open and closed subset of the connected space $\mathcal{P}$, must be all of $\mathcal{P}$.
Thus every problem has a solution, which finishes the existence part of
Alexandrov's argument, and with it his proof of the PL Minkowski
theorem.

Using the PL version just proved, together with a convergence argument,
Minkowski was then able to obtain the following more general result.
We incorporate Bonnesen and Fenchel’s observation [9] that Minkowski’s
original proof is easily extended to any dimension, by so stating the
theorem.

(1.8) MINKOWSKI’S THEOREM. Let $F$ be a nonnegative, completely
additive set function defined on all Borel subsets of the unit sphere $S^n \subset
R^{n+1}$, $n \geq 2$, such that

1. $\int_{S^n} \#F(d\Omega)=0$,
2. $F(S)<F(S^n)$ for every great $(n-1)$ sphere $S$ on $S^n$.

Then $F$ is the area function of a closed convex hypersurface $M^n \subset R^{n+1}$ which
is unique up to parallel translation.

Note that condition (1) is the analogue of the equation $\sum F_i \#i=0$ in
the PL version, while (2) is the analogue of the requirement that the
vectors $\#_1, \ldots, \#_m$ be noncoplanar. See [33].

What Minkowski really wanted to prove, but did not, was

(1.9) SMOOTH MINKOWSKI THEOREM. Let $K: S^n \to R^1$, $n \geq 2$, be a smooth
strictly positive function such that $\int_{S^n} (\#(p)|K(p)|d\Omega)=0$. Then there exists
a smooth closed convex hypersurface $M^n \subset R^{n+1}$, unique up to parallel
translation, whose Gauss map $\gamma: M^n \to S^n$ is a homeomorphism, such that
the Gaussian curvature of $M$ at the point $\gamma^{-1}(p)$ is $K(p)$ for all $p \in S^n$.

The most difficult thing was to prove the smoothness of the solution,
given a smooth preassigned curvature function $K$, and this was done
as follows.
The generalized Minkowski problem. A question more primitive than that answered by Minkowski's theorems is whether a preassigned strictly positive curvature function, unencumbered by any integrability condition, can be realized by some embedding of $S^n \rightarrow \mathbb{R}^{n+1}$, not necessarily the inverse of the Gauss map.

(2.1) Generalized Minkowski Theorem. Let $K: S^n \rightarrow \mathbb{R}^1$, $n \geq 2$, be a strictly positive function. Then there exists an embedding $G: S^n \rightarrow \mathbb{R}^{n+1}$ onto a closed convex hypersurface $M^n$, whose Gaussian curvature at the point $G(p)$ is $K(p)$ for all $p \in S^n$, subject to the following smoothness conditions:

For $n=2$: $K \in C^r \Rightarrow G \in C^{r+1}$ for $r=0$ or $r^2$.

For $n \geq 3$: $K \in C^r \Rightarrow G \in C^r$ for $r^3$.

The proof can be found in [17]. For $n=1$, this theorem has a variation in which the required embedding $G$ exists if and only if $K$ has at least two maxima and two minima, and one obtains thereby a converse to the classical 4-vertex theorem of differential geometry. A simplified proof, covering this case only, appears in [18].

(2.2) A Reduction of the Proof. Given the strictly positive function $K: S^n \rightarrow \mathbb{R}^1$ as in the hypotheses, one should first check in spirit whether or not $\int_{S^n} \left( \frac{\partial(h(p)/K(p))}{\partial q} \right) d\Omega^2 = 0$. If this integral vanishes, just apply the Smooth Minkowski Theorem (1.9) to get the result. But in general the integral does not vanish; its value is some random vector in $\mathbb{R}^{n+1}$.

Suppose for the moment that the problem is solvable, that is, that there is an embedding $G: S^n \rightarrow \mathbb{R}^{n+1}$ with the curvature of the image $M^n$ at the point $G(p)$ being $K(p)$. Then the curvature of the image at the point $\gamma^{-1}(q)$, where $\gamma: M^n \rightarrow S^n$ is the Gauss map, is $K(G^{-1}\gamma^{-1}(q))$. Hence by the integrability condition for (1.9)

$$\int_{S^n} \frac{\partial h(p)}{\partial q} d\Omega = 0,$$

where $h$ stands for the diffeomorphism $\gamma G$ of $S^n$ onto itself.

Conversely, if a diffeomorphism $h$ of $S^n$ onto itself exists making the above integral zero, let $M^n$ be the solution of the smooth Minkowski
problem for the curvature function $K r^{-1}$. Then define $G = \gamma^{-1} h$, and note that the curvature of $M^n$ at the point $G(p) = \gamma^{-1}(h(p))$ is $K r^{-1}(h(p)) = K(p)$.

(2.3) CONCLUSION. To solve the generalized Minkowski problem, it is necessary and sufficient to find a diffeomorphism $h$ of $S^n$ onto itself such that the above integral vanishes.

(2.4) REMARK. More generally, the following is true.

Let $M^n \subset \mathbb{R}^{n+1}, n \geq 2$, be a smooth closed manifold, $\theta(x)$ the unit outward normal vector to $M$ at $x$, and $f : M \rightarrow \mathbb{R}$ any continuous function. Then there exists a diffeomorphism $h$ of $M$ onto itself, diffeotopic to the identity, for which

$$\int_{M^n} f h^{-1}(x) \theta(x) \, dA = 0.$$  

This can also be found in [17].

(2.5) ANALOGY WITH A PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA. The hunt for such a diffeomorphism $h$ proceeds by analogy with the topologist's hunt for a root of the complex polynomial equation $f(z) = 0$. Here we are looking for a root of the equation

$$I(h) = \int_{M^n} f h^{-1}(x) \theta(x) \, dA = 0.$$  

One starts by regarding the group $\text{Diff}(M^n)$ of diffeomorphisms of $M$ onto itself with the $C^\infty$ topology as the analogue of the complex plane, but soon abandons this in favor of a certain $(2n+2)$-cell $B^{2n+2}$ in $\text{Diff}(M^n)$, which now shares with the complex plane the advantage of being contractible. A certain $n$-sphere $\Sigma^n \subset B^{2n+2}$ is constructed to play a role analogous to that of a circle of large radius in the complex plane. Specifically, no roots of the equation $I(h) = 0$ can be found on $\Sigma^n$. Thus

$$I|\Sigma^n : \Sigma^n \rightarrow \mathbb{R}^{n+1} = \{0\},$$  

and the final step is to prove, still faithful to the analogy, that this map is essential. The conclusion that $I$ has roots in $B^{2n+2}$ is then inescapable, and the argument ends.

3. Converse to the Gauss-Bonnet theorem for smooth manifolds. If the smooth closed two-manifold $M^2$ has a Riemannian metric with curvature $K$, then the beautiful theorem of Gauss [15] and Bonnet [10] relates this to the Euler characteristic:

$$\int_M K \, dA = 2\pi \chi(M).$$  

This in turn limits the possible functions which can serve as the curvature of a Riemannian metric on a given surface. For example, a negative function on the two-sphere could not possibly be the Gaussian curvature
of a Riemannian metric there because there is no way it could integrate to $4\pi$. This constraint, imposed on the curvature by the topology of the surface, is all the more remarkable in that no others are known.

(3.1) **The Sign Condition.** If $K$ is to be the Gaussian curvature of some Riemannian metric on $M$, we cannot formulate the Gauss-Bonnet theorem in advance, because before $M$ gets a Riemannian metric, it does not have an area element. We can extract, however, the following constraint on the sign of $K$:

(a) If $\chi(M)>0$, $K$ must be positive somewhere on $M$.
(b) If $\chi(M)=0$, $K$ (if not identically 0) must change sign.
(c) If $\chi(M)<0$, $K$ must be negative somewhere.

(3.2) **Three Related Problems.** The converse to the Gauss-Bonnet theorem can now be recorded as

**Conjecture A.** If $K$ satisfies the sign condition, then there is a Riemannian metric on $M$ having $K$ as its Gaussian curvature.

We see from the Generalized Minkowski Theorem (2.1) that this conjecture is true if $M$ is a two-sphere and $K>0$ everywhere. In the remaining cases, Kazdan and Warner found it advantageous to relate the problem to two further conjectures.

**Conjecture B.** If $K$ satisfies the sign condition, then there is a Riemannian metric on $M$, conformally equivalent to a preassigned metric, having $K$ as its Gaussian curvature.

The metrics $g$ and $g_0$ on $M$ are conformally equivalent if there is a smooth real valued function $u$ on $M$ and a diffeomorphism $\varphi$ of $M$ onto itself such that $g=\varphi^* (e^{2u}g_0)$, the pullback of $e^{2u}g_0$.

**Conjecture C.** If $K$ satisfies the sign condition, then there is a Riemannian metric on $M$, pointwise conformal to a preassigned metric, having $K$ as its Gaussian curvature.

The metrics $g$ and $g_0$ are pointwise conformal if there is a smooth real valued function $u$ on $M$ such that $g=e^{2u}g_0$.

Clearly, Conjecture C $\Rightarrow$ Conjecture B $\Rightarrow$ Conjecture A.

Here is a table of results.

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CONCLUSION. Conjecture C is true for the projective plane and false otherwise.

Conjecture B is true.

Conjecture A, the converse to the Gauss-Bonnet theorem for smooth closed two-manifolds, is true.

(3.3) THE EQUATION $\Delta_0 u = K_0 - K e^{2u}$. In order to hint at the details, we must first translate the condition $g = e^{2u}g_0$ of Conjecture C into a partial differential equation for the unknown function $u$ on $M$. Let $K_0$ be the Gaussian curvature of the original metric $g_0$, and $K$ the Gaussian curvature of the desired metric $g$.

To derive Equation (3.3) in which $\Delta_0$ denotes the Laplace operator for the metric $g_0$, write

$$g_0 = e^{2u(x,y)}(dx^2 + dy^2)$$

in local conformal parameters [13]. Then one computes:

$$K_0 = e^{-2u}(v_{xx} + v_{yy}), \quad \Delta_0 u = -e^{-2u}(u_{xx} + u_{yy}).$$

If we apply the above curvature formula to the metric

$$g = e^{2u}g_0 = e^{2(u+v)}(dx^2 + dy^2),$$

we get

$$K = e^{-2(u+v)}(u_{xx} + v_{xx} + u_{yy} + v_{yy}).$$

But then

$$K_0 - Ke^{2u} = -e^{-2u}(u_{xx} + u_{yy}) = \Delta_0 u,$$

as desired.

(3.4) INTEGRABILITY CONDITIONS FOR $\Delta_0 u = K_0 - K e^{2u}$. Roughly, the plan is first to try to solve Equation (3.3) for the unknown function $u$, which amounts to answering Conjecture C affirmatively. Failing this, one tries for integrability conditions for the equation. If they are not satisfied for the proposed curvature function $K$, one then looks for an appropriate diffeomorphism $\varphi: M^2 \to M^2$ so that they will be satisfied for the curvature function $K\varphi^{-1}$. In that case, $K\varphi^{-1}$ will be the curvature of a metric $e^{2u}g_0$ on $M^2$, and hence $K$ the curvature of the metric $\varphi^*(e^{2u}g_0)$, which will answer Conjecture B affirmatively.

Here are a few details.

(A) On the two-sphere $S^2$.

Koutroufiotis [29]. If $K$ is antipodally symmetric, $K(x) = K(-x)$, and close in measure to $K_0 \equiv 1$, then Equation (3.3) is integrable.

Moser [34]. If $K(x) = K(-x)$ and $K_0 \equiv 1$, the equation is integrable.

Kazdan and Warner [24]. In general, Equation (3.3) is not integrable. For example, suppose $g_0$ is the metric induced on the unit sphere $S^2 \subset R^3$ by the Euclidean metric of $R^3$, so that $K_0 \equiv 1$. Then a necessary condition
which \( K \) must satisfy in order that (3.3) be integrable is

\[
\int_{S^2} e^{2u} (\nabla_0 K) \cdot (\nabla_0 F) \, dA_0 = 0,
\]

for all first order spherical harmonics \( F \), where \( \nabla_0 K \) and \( \nabla_0 F \) denote the gradient vector fields of the functions \( K \) and \( F \) with respect to the round metric \( g_0 \), the dot product is with respect to \( g_0 \), and \( dA_0 \) is the area element of \( g_0 \). Now the coordinate functions \( x, y, z \) are the first order spherical harmonics on \( S^2 \) in the standard metric. If \( K = z + 2 \), for example, so that it is strictly positive, and if \( F = z \), then the above integral is also strictly positive, so that the integrability condition is not satisfied. Hence \( K = z + 2 \) cannot be the curvature of a metric on \( S^2 \) pointwise conformal to the standard metric \( g_0 \). Similar integrability conditions are obtained by Kazdan and Warner for any initial metric \( g_0 \).

(B) \textit{On the projective plane} \( P^2 \).

Moser [34]. If \( K_0 \equiv 1 \), then Equation (3.3) is always integrable on the projective plane, and an extension of Moser's techniques yields the same result for any initial metric \( g_0 \).

(C) \textit{On the torus and Klein bottle}.

Kazdan and Warner [24]. Suppose \( g_0 \) is a flat metric, so that \( K_0 \equiv 0 \). If \( K \) is identically 0, the problem is trivial; otherwise we know that \( K \) takes both positive and negative values by the sign condition. In this case, Kazdan and Warner obtain a necessary and sufficient condition for the integrability of (3.3), namely

\[
\int_{M^2} K \, dA_0 < 0.
\]

Note the hybrid nature of this condition. It looks suspiciously like the Gauss-Bonnet theorem, except that equality is replaced by inequality. The point is that \( K \) is the curvature of the desired metric \( g = e^{2u} g_0 \), while \( dA_0 \) is the area element of the original metric \( g_0 \).

This is a good place to see the general plan in action. If the above integrability condition is not satisfied by the proposed curvature function \( K \), first find a point \( p \) in \( M^2 \) at which \( K(p) < 0 \). Then find a diffeomorphism \( \varphi \) of \( M^2 \) onto itself which spreads a small neighborhood \( U \) of \( p \), on which \( K < 0 \), over most of \( M^2 \). Then \( K \varphi^{-1} < 0 \) on most of \( M^2 \), so that the integrability condition \( \int_{M^2} K \varphi^{-1} \, dA_0 < 0 \) is certainly satisfied. Thus \( K \varphi^{-1} \) is the curvature of a metric \( e^{2u} g_0 \) on \( M^2 \), and, hence, \( K \) is the curvature of the metric \( \varphi^*(e^{2u} g_0) \). Therefore Conjecture B is answered affirmatively in this case. Similar results are obtained for any initial metric \( g_0 \).

(D) \textit{On surfaces with} \( \chi < 0 \).
Poincaré [40] and (by a different method) Mel Berger [8]. If $K<0$ everywhere on $M^2$, then Equation (3.3) is integrable for any initial metric $g_0$.

Kazdan and Warner [24]. Suppose $g_0$ is a metric of constant negative Gaussian curvature. Then a necessary condition for the solvability of (3.3) is that

$$\int_{M^2} K dA_0 < 0,$$

just as for the torus and Klein bottle, except that now the condition is no longer sufficient. Similar results hold for any initial metric $g_0$.

(3.5) It is clear from the above remarks that the treatment of Equation (3.3), and hence of Conjectures B and C, depends heavily on the topology of the surface $M^2$. Here is one more item in that direction.

Rewrite Equation (3.3) as

$$T(u) = -e^{-2u}(\Delta_0 u - K_0) = K,$$

and note that $T(0)=K_0$. It is natural to linearize this map $T$, because if the linearized version is invertible, then one can apply the inverse function theorem for Banach spaces to solve Equation (3.3) for $K$ sufficiently close to $K_0$. The curious result is that the linearized version is invertible for the projective plane with the standard metric and for surfaces of strictly negative Euler characteristic, but not invertible for certain initial metrics on the two-sphere, torus and Klein bottle (Kazdan and Warner [27], Koutsoufiotis [29]).

(3.6) For a uniform overview of the attack on Conjectures B and C, I suggest Kazdan and Warner [27]. For a somewhat different approach to Conjecture A, without the intermediacy of Conjectures B and C, see Kazdan and Warner [28], which depends on a result of Bourguignon [11] and Fisher and Marsden [14].

(3.7) NONCOMPACT SURFACES. Let $M^2$ be a noncompact surface, diffeomorphic to an open subset of some compact surface. Then any $C^\infty$ function is the Gaussian curvature of some Riemannian metric on $M^2$.

See Kazdan and Warner [25].

4. Converse to the Gauss-Bonnet theorem for PL manifolds. Now let $M^2$ be a compact PL two-manifold, possibly with boundary. A PL Riemannian metric on $M$ is determined by a triangulation of $M$ and by linear metrics on each two-simplex of the triangulation, consistent with respect to the face operation. The curvature at an interior vertex is $2\pi$ minus the sum of the angles around that vertex; the exterior angle at a boundary vertex is $\pi$ minus the angle sum.
The elementary geometry of Riemannian polyhedra unfolds in a manner similar to that for smooth manifolds. For details in this direction, including a discussion of curvature in the PL category, I suggest [6], [7], [19] and [42].

In any case, the Gauss-Bonnet theorem holds for compact PL surfaces, and reads:

$$\sum_{M} \text{curvature} + \sum_{\partial M} \text{exterior angles} = 2\pi \chi(M).$$

Since curvature here is the analogue of integral curvature in the smooth category, this equation acts, undiluted, as a constraint on any such preassigned data.

(4.1) Theorem (Joint with Kenneth Krigelman and David Singer). Let \( M \) be a connected compact PL two-manifold possibly with boundary. Let \( p_1, \ldots, p_r \) be points in the interior \( \bar{M} \) of \( M \). Let \( q_1, \ldots, q_s \) be points on the boundary \( \partial M \) of \( M \).

Given the data: (1) real numbers \( k_1, \ldots, k_r \), each <2\(\pi\),

(2) real numbers \( e_1, \ldots, e_s \), each <\(\pi\), subject to the constraint:

$$\sum_{i} k_i + \sum_{j} e_j = 2\pi \chi(M).$$

Then there exists a PL Riemannian metric on \( M \) with curvatures \( k_i \) at each \( p_i \), exterior angles \( e_j \) at each \( q_j \), and flat elsewhere.

The proof is combinatorial, somewhat reminiscent of the classification procedure for compact triangulable surfaces. It was first obtained by Singer for the two-sphere with positive preassigned curvature, then for the two-sphere without this restriction by the author, then by Krigelman (Ph.D. thesis, University of Pennsylvania, 1972) for closed surfaces, and finally in the above form. The reader is referred to [19] for details.

5. Realization in three-space. In both the smooth and PL versions of the converse to the Gauss-Bonnet theorem, the desired Riemannian metric was produced in abstracto, not via any specific model in Euclidean space. One can of course add this as an extra requirement, and again ask for the answer.

In the smooth case there is no problem because, by the Nash embedding theorem [36], a Riemannian metric on a compact manifold can always be realized by a model in Euclidean space.

In the PL case, the analogue of the Nash embedding theorem is yet to be proved. Even the specific Riemannian metrics obtained in [19] to exhibit preassigned curvature data are not known to be realizable in Euclidean space.
Consider, by contrast, the question of realizability of solutions in 3-space. In general, of course, the answer is no. A smooth closed surface in 3-space must have points of strictly positive curvature. Hence a metric with Gaussian curvature everywhere negative on a closed surface of negative Euler characteristic cannot be realized in 3-space. And the closed nonorientable surfaces cannot even be realized in 3-space for topological reasons. Similarly in the PL category.

One case is a natural for the extra requirement of embeddability in 3-space, namely that of positive preassigned curvature on a manifold homeomorphic to the two-sphere. In the smooth case one gets this for free.

The Weyl problem starts with a Riemannian metric of strictly positive curvature on a smooth manifold homeomorphic to the two-sphere, and asks for a realization by a smooth (necessarily convex) surface in 3-space. This problem was posed and partially solved by Weyl in 1916 [43], complete solutions being given by Pogorelov [38] and Nirenberg [37]. As a consequence, once a strictly positive curvature function on a space homeomorphic to the two-sphere is realized by some Riemannian manifold in abstracto, one just appeals to the solution of the Weyl problem to embed it isometrically in 3-space.

In the PL case, the corresponding argument misses. The PL version of the Weyl problem starts with a PL Riemannian metric of positive curvature on a manifold homeomorphic to the two-sphere. One cannot possibly begin with strictly positive curvature, because all but finitely many points must have zero curvature. The affirmative solution was given by Alexandrov [6]. But it has one degenerate case. A doubly covered plane region, bounded by a convex polygon, is considered a degenerate case of a closed convex surface in 3-space. This degenerate case is unavoidable because it exists, and because the corresponding uniqueness result (the Cauchy rigidity theorem) applies to it and prevents an alternative nondegenerate realization.

Unfortunately, the solution given in [19] to the converse of the PL Gauss-Bonnet theorem lands one exactly on the degenerate case of the Weyl problem. The uniqueness aspect of the Weyl problem does not prevent an alternative realization as a nondegenerate closed convex surface in 3-space, since here we are interested in realizing only preassigned curvature, not preassigned metric.

An ingenious argument has been given by David Singer [41].

(5.1) Theorem (David Singer). Let $p_1, \ldots, p_r$ be points on a PL two-sphere $S$ and $k_1, \ldots, k_r$ real numbers with $0 < k_i < 2\pi$ and $\sum k_i = 4\pi$. Then there exists a PL embedding of $S$ into $\mathbb{R}^3$ with convex image, realizing these preassigned curvatures $k_i$ at the points $p_i$ and flat elsewhere.
(5.2) **Tetrahedra with Preassigned Curvature.** In order to carry out the inductive step of Singer’s argument, one needs to know the existence of tetrahedra with preassigned curvature and base.

Starting for example with a triangle $V_1V_2V_3$ in 3-space, one wants to extend it to a tetrahedron $V_1V_2V_3V_4$ with curvatures $k_1, k_2, k_3, k_4$ at its four vertices, where $0<k_i<2\pi$ and $\sum_i k_i=4\pi$. If the angles of $V_1V_2V_3$ are denoted by $a_1, a_2, a_3$, then one must assume $k_i<2\pi-2a_i$ or a solution will be impossible. Given this data, however, the desired tetrahedron exists and is unique up to reflection through its base.

The argument, though not difficult, reflects the charm of Alexandrov’s methods. Parametrize the space of all tetrahedra with base $V_1V_2V_3$ by the numbers $x_1, x_2, x_3$ which represent the distance from $F$ to $V_1, V_2, V_3$, respectively. We are therefore not distinguishing between a tetrahedron and its reflection through the base. The curvatures $k_1, k_2, k_3$ depend differentiably on $x_1, x_2, x_3$ and one may compute the Jacobian determinant of this dependence. Somewhat involved at first, the computation yields the explicit answer:

$$\frac{\partial (k_1, k_2, k_3)}{\partial (x_1, x_2, x_3)} = 2 \frac{\cos(\alpha + \beta + \gamma) - 1}{x_1x_2x_3 \sin \alpha \sin \beta \sin \gamma},$$

where $\alpha, \beta, \gamma$ are the angles at $V_4$ of the three faces. Since $0<\alpha+\beta+\gamma<2\pi$, this Jacobian is always $<0$. If one regards the set of parameters $\{(x_1, x_2, x_3)\}$ as the solution space $\mathcal{S}$, and the set of allowable curvatures $\{(k_1, k_2, k_3)\}$ as the problem space $\mathcal{P}$, then the natural map $\varphi: \mathcal{S} \rightarrow \mathcal{P}$ has nonsingular Jacobian everywhere. Both $\mathcal{S}$ and $\mathcal{P}$ are homeomorphic to open 3-cells, so $\varphi$ must be an open map.

To see that $\varphi$ is also a closed map, Singer argues as follows. Compactify the solution space $\mathcal{S}$ by adding “idealized solutions”, essentially degenerate tetrahedra with control on the angles at each vertex, to form a space $\overline{\mathcal{S}}$ homeomorphic to a closed 3-cell. Similarly, compactify the problem space $\mathcal{P}$ by adding degenerate problems, that is, allowing $0\leq k_i \leq 2\pi - 2a_i$, to form a space $\overline{\mathcal{P}}$ also homeomorphic to a closed 3-cell. The map $\varphi$ extends to a map $\overline{\varphi}: \overline{\mathcal{S}} \rightarrow \overline{\mathcal{P}}$, which takes $\overline{\mathcal{S}} - \mathcal{S}$ to $\overline{\mathcal{P}} - \mathcal{P}$. In other words, degenerate solutions correspond to degenerate problems. But the map $\overline{\varphi}$ is closed because $\overline{\mathcal{S}}$ is compact. It follows immediately that $\varphi$ is closed.

Now $\varphi(\mathcal{S})$ is a nonempty open and closed subset of the connected space $\overline{\mathcal{P}}$, hence all of $\overline{\mathcal{P}}$, and therefore every problem has a solution.

(5.3) **Remainder of Singer’s Proof.** Using this lemma about tetrahedra, Singer constructs a surface bounding a polyhedral 3-cell in $R^{t-1}$ and having preassigned strictly positive curvatures $k_1, \cdots, k_r$. The argument is
by induction on \( r \), so that the 3-cell is constructed by adding an appropriate tetrahedron to a polyhedral 3-cell in \( \mathbb{R}^{r-2} \), which was previously obtained as a solution to a corresponding problem with only \( r-1 \) preassigned curvatures. The step in dimensions from \( r-2 \) to \( r-1 \) is there to prevent self-intersections: if the construction is carried out in \( \mathbb{R}^3 \), one cannot prove convexity directly by this method, and therefore cannot avoid self-intersections during the inductive step.

The final part of the argument is a surprise. One has just produced by the above method a polyhedral two-sphere with preassigned curvatures \( k_1, \ldots, k_r \), but it lies in \( \mathbb{R}^{r-1} \) rather than in \( \mathbb{R}^3 \). Now comes an application of Alexandrov's solution of the Weyl problem to produce an isometric copy of this two-sphere as a convex polyhedral surface in 3-space. The problem is to avoid the degenerate case.

Notice that in a degenerate solution to the Weyl problem, namely a doubly covered plane convex polygonal region, each vertex of positive curvature can be joined to its immediate neighbor on either side by a unique path of shortest length on the surface. But to every other vertex of positive curvature it has exactly two shortest paths on the surface, one on each sheet.

Compare this with the two-sphere bounding the 3-cell in \( \mathbb{R}^{r-1} \), which in turn was obtained by adding a tetrahedron to a 3-cell in \( \mathbb{R}^{r-2} \). The vertex of that tetrahedron is a point of positive curvature on the resulting two-sphere, and it is joined by straight line paths in \( \mathbb{R}^{r-1} \) to the three vertices at its base, all of which represent points of positive curvature on the two-sphere. Because the model is in Euclidean space, there can be no other paths of that same shortest length in the ambient space, let alone on the two-sphere.

It follows that this two-sphere cannot be isometric to any degenerate solution of the Weyl problem. But it is isometric to some solution because it has positive curvature, so that the solution must be nondegenerate and therefore furnishes the required convex two-sphere in 3-space with the preassigned curvature.

(5.4) Robert Connelly has found, by an entirely different method, an explicit construction of a convex polyhedral two-sphere circumscribed about the unit sphere in 3-space and having preassigned positive curvature.

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