ROUND HANDLES AND HOMOTOPY OF NONSINGULAR VECTOR FIELDS

BY DANIEL ASIMOV

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Introduction. We consider nonsingular vector fields on compact connected $C^\infty$ manifolds. The question is: What orbit structures occur in which homotopy classes of nonsingular vector fields? We show that in dimensions 4 and greater, a nonsingular Morse-Smale (NMS) vector field occurs in each homotopy class. Under the additional assumption that the first Betti number of the manifold is nonzero, we show that a nonsingular volume-preserving (NVP) vector field occurs in each homotopy class. These results are based on the round handle decomposition theorem, interesting in its own right as a structure theorem for manifolds whose Euler characteristic is 0.

Let $M$ be a compact manifold whose boundary has been partitioned into two unions of components: $\partial M = \partial_- M \cup \partial_+ M$, $\partial_- M \cap \partial_+ M = \emptyset$. Then the following are equivalent:

1. $\chi(\partial_- M) = \chi(M)$.
2. $\chi(\partial_+ M) = \chi(M)$.
3. There exists a nonsingular vector field on $M$ pointing inward on $\partial_- M$ and outward on $\partial_+ M$.

DEFINITION. The pair $(M, \partial_- M)$ will be called a flow manifold if 1, 2, and 3 above are true. This does not exclude the possibility, of course, that $\partial_- M$, $\partial_+ M$, or $\partial M$ may be empty.

DEFINITION. A nonsingular Morse-Smale (NMS) vector field $V$ on the flow manifold $(M, \partial_- M)$ is one which satisfies (a), (b) and (c) below:

(a) $V$ has nonwandering set equal to a finite number of closed orbits, each having a hyperbolic Poincaré map.

(b) The stable manifold (inset) of one closed orbit is transversal to the unstable manifold (outset) of any other closed orbit.

(c) $V$ points inward on $\partial_- M$ and outward on $\partial_+ M$.

DEFINITIONS. A round handle of index $k$ (and dimension $n$) is a copy
of $R_k = S^1 \times D^k \times D^{n-k-1}$. The attaching region $\partial_- R_k$ of $R_k$ is the submanifold $S^1 \times S^{k-1} \times D^{n-k-1}$ of $\partial R_k$, where $S^{k-1} = \partial D^k$.

**Theorem 1 (Round Handle Decomposition Theorem).** Let $(M^n, \partial_- M)$ be a flow manifold with $n \geq 4$. Then $M$ admits a decomposition of the form

$$M = (\partial_- M \times I) + R_0^1 + \cdots + R_0^\alpha + R_1^1 + \cdots + R_{n-1}^\alpha.$$

This means that each $R_k^l$ is attached via a diffeomorphism of $\partial_- R_k$ to the boundary of the previous stuff, but never to $\partial_- M \times \{0\}$. If we further assume that $\partial_- M \neq \emptyset$ and $\partial_+ M \neq \emptyset$, we may write

$$M = (\partial_- M \times I) + R_1^1 + \cdots + R_{n-2}^\beta,$$

i.e. there exists a round handle decomposition avoiding round handles of extreme indices.

**Corollary 1.** Let $(M^n, \partial_- M)$ be a flow manifold with $n \geq 4$. Then $(M, \partial_- M)$ admits an NMS vector field. If we further assume that $\partial_- M \neq \emptyset$ and $\partial_+ M \neq \emptyset$, then $(M, \partial_- M)$ possesses an NMS vector field with no source or sink closed orbits.

**Theorem 2.** Let $V$ be a nonsingular vector field on the flow manifold $(M^n, \partial_- M)$ pointing inward on $\partial_- M$ and outward on $\partial_+ M$. Assume $M$ is orientable and $n \geq 4$. Then $V$ is homotopic rel $\partial M$ (through nonsingular vector fields) to an NMS vector field. If further $\partial_- M \neq \emptyset$ and $\partial_+ M \neq \emptyset$ then $V$ is homotopic rel $\partial M$ to an NMS vector field with no sources or sinks.

**Theorem 3.** Let $M^n$ be compact, connected, and orientable with $\chi(M) = 0$, $n \geq 4$, and $H^1(M) \neq 0$. Then any nonsingular vector field $V$ on $M$ is homotopic to a field $V_1$ which preserves some smooth nonzero volume form $\omega$ on $M$ (which may be preassigned).

**Proof (Sketch).** Using $H^1(M) \neq 0$ we first find a compact, connected, oriented submanifold $i: N^{n-1} \subseteq M$ with $i_*[N] \neq 0$ in $H_{n-1}(M)$. By surgering $N$ we may find a homologous submanifold $N'$ and a vector field $V'$ homotopic to $V$ such that $V'$ is nowhere tangent to $N'$. Then cutting $M$ open along $N'$ gives us a flow manifold $\tilde{M}$, and $V'$ becomes a vector field $\tilde{V}$ on $\tilde{M}$. By Theorem 2, $\tilde{V}$ is homotopic rel $\partial \tilde{M}$ to an NMS vector field $\tilde{V}_1$ on $\tilde{M}$ having no sources or sinks. This last property permits finding (one round handle at a time) a smooth nonzero volume $\tilde{\omega}$ on $\tilde{M}$ preserved by $\tilde{V}_1$. By carefully regluing
\( \tilde{M} \) to obtain \( M \) once again, \( \tilde{V}_1 \) becomes a smooth nonsingular vector field \( V_1 \) (which is not NMS), and \( \tilde{\omega} \) becomes a smooth nonzero volume \( \omega \) on \( M \) preserved by \( V_1 \). The form \( \omega \) may as well have been preassigned, since Moser [4] shows any two volume forms are equivalent up to constant multiple under a diffeomorphism of \( M \) isotopic to the identity.

Remark. Theorem 3 fails on the 2-torus \( T^2 \).

REFERENCES

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455