A RADIAL MULTIPLIER AND
A RELATED KAKEYA MAXIMAL FUNCTION

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In this paper we state some results for a maximal function and a Fourier multiplier that are connected with the Bochner-Riesz spherical summation of multiple Fourier series (see Fefferman [3], [5]). Our purpose will be to get sharp estimates for the norm of these operators in dimension two. Proofs will appear elsewhere [2].

Let \( N \geq 1 \) be a real number. By a rectangle of eccentricity \( N \) we mean a rectangle \( R \) such that

\[
\frac{\text{Length of the bigger side of } R}{\text{Length of the smaller side of } R} = N.
\]

We will define the direction of \( R \) as the direction of its bigger side.

Given a locally integrable function \( f \) we consider the maximal function

\[
Mf(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)| \, dy,
\]

where the "Sup" is taken over rectangles of eccentricity \( N \), but arbitrary direction.

**Theorem 1.** The sublinear operator \( M \) is bounded in \( L^2(\mathbb{R}^2) \) and there exists a constant \( C \), independent of \( N \), such that

\[
\||Mf||_2 \leq C(\log 3N)^2 \|f\|_2.
\]

Suppose that \( m_0 \) is a smooth function on \( R \) with support on \((-1, 1)\) and let \( m(r) = m_0(\delta^{-1}(r - 1)) \), where \( \delta > 0 \) is a small number.

Consider the Fourier multiplier defined by

\[
\hat{Mf}(\xi) = m(|\xi|)\hat{f}(\xi), \quad f \in C_c^\infty(\mathbb{R}^2).
\]

**Theorem 2.** There exists a constant \( C \), independent of \( \delta \), such that...
\[ \|Tf\|_4 \leq C \log \delta^{5/4} \|f\|_4, \quad \forall f \in C_0^\infty (\mathbb{R}^2). \]

Our proofs of Theorems 1 and 2 are made in the spirit of Cotlar's lemma. In particular the support of the kernel of \( T \) can be decomposed into a family of rectangles of eccentricity \( \delta^{-1/4} \) and the convolution operators, obtained by restricting the kernel to these rectangles, are "almost orthogonal".

Theorem 2 can be applied to get Bochner-Riesz summability below the critical index, in dimension two.

**Corollary 3 (Carleson-Sjolin-Fefferman-Hörmander).** The operator \( T_{\lambda} \) defined by \( \hat{T}_{\lambda}(\xi) = m_{\lambda}(\xi)\hat{\varphi}(\xi) \), where \( m_{\lambda}(\xi) = (1 - |\xi|^2)^{\lambda} \) if \( |\xi| \leq 1 \) and \( m_{\lambda}(\xi) = 0 \) otherwise, is bounded in \( L^p(\mathbb{R}^2) \) if

\[ \frac{4}{3 + 2\lambda} < p < \frac{4}{1 - 2\lambda}, \quad \frac{1}{2} > \lambda > 0. \]

To see this we define a partition of unity on \((0, 1)\) as follows: For every \( n, h_n \) is a smooth function with support on \((1 - 2^{-n+1}, 1 - 2^{-n-1})\) such that \( |D^p h_n(r)| \leq A_p 2^{np} \) (with \( A_p \) independent of \( n \)) and \( \sum h_n(r) = 1 \) on \((0, 1)\). Then \( m_{\lambda}(\xi) = \sum m_{\lambda}(\xi) h_n(|\xi|) \). If we apply Theorem 2 to the operator \( T_{\lambda}^n \) defined by the multiplier \( m_{\lambda}(\xi) h_n(|\xi|) \) we get that \( \|T_{\lambda}^n f\|_4 \leq C 2^{-n\lambda} n^{5/4} \|f\|_4 \). And then, Corollary 3 can be deduced from this estimate by standard arguments of interpolation, duality and adding a geometric series.

**Remark.** Theorem 2 can be used to prove a sharper version of Corollary 3 i.e., suppose that \( m \) is a smooth function on \((0, 1)\) such that it behaves like

\[ \left( \log \frac{1}{1 - |x|} \right)^{-t} \text{ near } |x| = 1. \]

Then \( m \) is a multiplier for \( L^p(\mathbb{R}^2) \), \( 4/3 < p < 4 \) provided that \( t > 9/4 \).

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**References**


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