Let \( \Lambda = \{\lambda_k\}_{k=1}^{\infty} \) be a sequence of distinct nonnegative real numbers. It is well known that the exponential sums

\[
e_s(x) = \sum_{k=1}^{\infty} a_k e^{\lambda_k t}, \quad a_k \in R, \quad s = 1, 2, \ldots,
\]

are dense in \( C[A, B] \), \( -\infty < A < B < +\infty \), if and only if Müntz' condition

\[
\sum_{k=1}^{\infty} 1/\lambda_k = +\infty
\]

holds. In this note Jackson-type results on the rate of convergence of the exponential sums (1) are given. Substituting

\[
x = e^{t-B}, \quad t \in [A, B], \quad x \in [a, 1],
\]

where \( a = e^{A-B} \), we are led to the problem where the functions \( f \in C[a, 1] \), \( 0 < a < 1 \), are to be approximated on \( [a, 1] \) by the \( \Lambda \)-polynomials

\[
p_s(x) = \sum_{k=1}^{s} b_k x^{\lambda_k}, \quad b_k \in R, \quad s = 1, 2, \ldots.
\]

Recently, many optimal or almost optimal Jackson-Müntz theorems on the approximation properties of the \( \Lambda \)-polynomials (3) for the interval \([0, 1]\) have been published (cf. J. Bak and D. J. Newman [1] and M. v. Golitschek [2]). Considering intervals \([a, 1]\), \( a > 0 \), one would expect that the \( \Lambda \)-polynomials have even better approximation properties than on \([0, 1]\), as the "singular" point \( x = 0 \) might have less influence. Theorems 1 and 2 prove this conjecture.

**Theorem 1.** Let \( 0 < a < 1, M > 0 \). If \( \Lambda \) satisfies

\[
0 < \lambda_k \leq Mk \quad \text{for all} \quad k = 1, 2, \ldots,
\]

then for each function \( f \in C^r[a, 1] \), \( r \geq 0 \), and each integer \( s \geq r + 1 \) there exists a \( \Lambda \)-polynomial \( p_s \) such that for all \( a \leq x \leq 1 \).
(5) \(|f(x) - p_s(x)| \leq K_r s^{-r} \omega(f^{(r)}; 1/s) + O(\rho^s)\),

where \(\omega\) denotes the modulus of continuity; \(K_r > 0\) depends on \(a, M,\) and \(r;\) and \(\rho\) \((0 < \rho < 1)\) depends only on \(a\) and \(M.\)

Consequently, if the exponents \(\Lambda\) satisfy (4), the \(\Lambda\)-polynomials behave asymptotically as well as the ordinary algebraic polynomials. As the \(s\)th width \(d_s(\Lambda_{r\omega})\) of the class \(\Lambda_{r\omega}(M_0, \cdots, M_{r+1}; [a, 1])\) of functions in \(C[a, 1]\) is

\[
d_s(\Lambda_{r\omega}) \approx s^{-r}\omega(1/s),
\]

(cf. G. G. Lorentz [3, Chapters 3.7 and 9.2]), the \(\Lambda\)-polynomials of Theorem 1 approximate asymptotically optimally in this special sense.

**EXAMPLE.** The exponents \(\Lambda\) with \(\lim_{k \to \infty} \lambda_k = \lambda > 0\) satisfy condition (4). For the corresponding problem in \([0, 1]\) we could only prove (cf. M. v. Golitschek [2, p. 95]) that there exist \(\Lambda\)-polynomials \(p_s\) for which

\[
|f(x) - p_s(x)| = O(\sqrt{s}^{-r}\omega(f^{(r)}; 1/\sqrt{s})), \quad s \to \infty.
\]

**THEOREM 2.** Let \(0 < a < 1, M > 0, e > 0.\) Let \(\Lambda\) satisfy

(6) \(\lambda_k \geq Mk\) for all \(k = 1, 2, \cdots.\)

For each \(s \geq s_0\) \((s_0\) sufficiently large) let \(\psi(s)\) be defined as the largest positive integer for which

(7) \(\sum_{\psi < k \leq s} \frac{1}{\lambda_k} \geq -(1 + e) \log \sqrt{a}.
\)

Then for each \(f \in C^r[a, 1]\) and each \(s \geq s_0\) there exists a \(\Lambda\)-polynomial \(p_s\) such that for all \(a < x < 1\)

(8) \(|f(x) - p_s(x)| \leq K_r \psi(s)^{-r} \omega(f^{(r)}; 1/\psi(s)) + O(\rho^s),\)

where \(K_r\) depends on \(a, r, M,\) and \(e;\) and \(\rho\) \((0 < \rho < 1)\) depends on \(a, M,\) and \(e.\)

**EXAMPLE.** Let \(\lambda_k = k \log k, k = 1, 2, \cdots, M = 1, e > 0.\) From (7) we obtain

\[
\psi(s) \approx s^{1+e}.\]

In [1] and [2] it was proved that in \([0, 1]\) the corresponding “rate of convergence” is only

\[
\varphi(s) = \exp \left( -2 \sum_{k=1}^{s} \frac{1}{k \log k} \right) \approx (\log s)^{-2}.
\]
The above theorems are proved by the same method used by the author in his earlier paper [2] for Jackson-Müntz theorems on the interval [0, 1]: First the function \( f \) is approximated by ordinary algebraic polynomials \( P_n \) and then each monomial \( x^q \) \((q = 0, 1, \ldots, n)\) of \( P_n \) is approximated by appropriate \( \Lambda \)-polynomials. The full details and further results will be published later.

By the substitution \( t = B + \log x \) we obtain from Theorems 1 and 2 immediately the corresponding approximation theorem for the exponential sums (1).

**THEOREM 3.** Let \( F \in C^r[A, B], -\infty < A < B < +\infty, r \geq 0 \). Let the best approximation of \( F \) be defined by

\[
E_s^*(F; \Lambda) = \inf_{a_k \in \Lambda} \max_{A < t < B} \left| F(t) - \sum_{k=1}^{s} a_k e^{\lambda_k t} \right|.
\]

If \( \Lambda \) satisfies (4), then

\[
E_s^*(F; \Lambda) = O(s^{-r} \omega(F^{(r)}; 1/s)) \quad \text{for} \quad s \to \infty.
\]

If \( \Lambda \) satisfies (6), then for each \( \epsilon > 0 \)

\[
E_s^*(F; \Lambda) = O(\psi(s)^{-r} \omega(F^{(r)}; 1/\psi(s))) \quad \text{for} \quad s \to \infty,
\]

where \( \psi(s) \) is defined by (7) with \( \log \sqrt{a} = (A - B)/2 \).

**REMARK.** The same results are also valid in the \( L_p \) norms, \( 1 \leq p < \infty \), if the function \( f \) (or \( F \)) has an \((r - 1)\)st absolutely continuous derivative in \([a, 1]\) (or \([A, B]\)) and \( f^{(r)} \in L_p(a, 1) \) (or \( F^{(r)} \in L_p(A, B) \)) and if \( \omega \) denotes the integral modulus of continuity in \( L_p \).

**BIBLIOGRAPHY**

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