THE DETERMINANT OF A RANDOM MATRIX

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An experimental worker measures $n \times n$ independent magnitudes $a_{ij}$ ($i, j = 1, \cdots, n$) and computes the determinant of the corresponding matrix before discarding the $a_{ij}$'s. One can imagine a set-up where he only gets to know the determinant but not the $a_{ij}$ themselves. By repeating the same experiment over and over and averaging the corresponding determinants he obtains the determinant of the unknown mean values $m_{ij}$ of the magnitudes $a_{ij}$.

The purpose of this note is to indicate that under reasonable assumptions he can get much more information about the unknown matrix $m_{ij}$ at no extra cost.

Assume that the entries $a_{ij}$ ($i, j = 1, \cdots, n$) are jointly Gaussian variables with unknown means $m_{ij}$ and known correlation matrix $R$.

For the remainder of this note we concentrate on the special case where the entries are independent and have the same variance, i.e. $R = \rho I$ ($\rho \neq 0$); this case already well illustrates our point. The results given here illustrate the fact that in many nonlinear identification problems the presence of noise can prove helpful (see [1]). We give first a complete analysis of the $2 \times 2$ case.

For the discussion below put

$$M(x) = \begin{pmatrix} a + x_1 & b + x_2 \\ c + x_3 & d + x_4 \end{pmatrix}$$

with $a, b, c, d$ unknown constants, and $x_1, x_2, x_3, x_4$ Gaussian $(0, 1)$ random variables. The case $(0, \rho)$ ($\rho \neq 0$), is just the same.

Bring in the characteristic function of the determinant $F(\lambda) = E e^{i\lambda \det M}$ and conclude after some elementary computation that

$$F(\lambda) = e^{i\lambda M(0)} \Psi(a + d, -\lambda) \Psi(d - a, \lambda) \Psi(b + c, -\lambda) \Psi(b - c, \lambda).$$

Here


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\[ \Psi(\alpha, \lambda) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-i\lambda)} e^{i\lambda \alpha \xi} \frac{d\xi}{\sqrt{2\pi}} = e^{-\frac{1}{2}(1-i\lambda)^{-1} \lambda^2 \alpha^2} \]

so that one finally gets

\[ F(\lambda) = \frac{e^{i\lambda \det M(0)}}{(1-i\lambda)^2} \times e^{-\lambda^2 \left( a^2 + b^2 + c^2 + d^2 \right) + i\lambda 2(ad - bc)} \]

In conclusion, we see that the knowledge of the distribution function of \( \det M(x) \) is equivalent to that of \( \det M(0) = ad - bc \), and \( \text{tr} MM^* = a^2 + b^2 + c^2 + d^2 \). Moreover it is plain that this information is already contained in the first two moments of \( \det M(x) \). This is summed up in

**Theorem I.** From the distribution function of \( \det M(x) \) one obtains exactly \( \det M(0) \) and \( \text{tr} M(0)M^*(0) \).

**Corollary.** If the matrix of mean values \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is symmetric, then from the distribution of \( \det M(x) \) one can deduce both eigenvalues of \( M(0) \) up to a common change in sign.

**Remark.** If \( a_{12} \) and \( a_{21} \) had zero variances, and only \( a_{11}, a_{22} \) were subject to a random error of the same nonzero variance, our proof shows that \( a_{11} \) and \( a_{22} \) could be determined up to order and a common change in sign. Moreover \( \det M(0) \) is known, and thus we also have the product \( a_{12}a_{21} \).

Notice that we have found the determinant and the trace of \( M(0)M^*(0) \), besides \( \det M(0) \). It is in this form that the results above extend most naturally to higher dimensions. We have

**Theorem II.** The distribution function of \( \det M(x) \) determines exactly the eigenvalues of \( M(0)M^*(0) \)—alias the singular values of \( M(0) \)—and \( \det M(0) \).

The proofs are not so simple as for \( n = 2 \). There is no closed form for the characteristic function of \( \det M(x) \) and one has to compute moments and then solve the resulting system of equations. Some indications of how this is done will be given below.

If each entry has (known) variance \( \rho \) one shows that

\[ E \det M(x) = \det M(0) \equiv \det M, \]

\[ E(\det M(x))^2 = (\det M)^2 + \rho \Sigma(M^{ij})^2 + 2\rho^2 \Sigma(M^{ijkl})^2 + \cdots + n!\rho^n. \]

On the last line the first sum runs over the square of all the \((n-1) \times (n-1)\) minors of \( M \), the second one over all the \((n-2) \times (n-2)\) minors (squared) of \( M \), and so on. This formula is all that is needed if the variance \( \rho \) is variable;
see below. The expressions for the higher moments involve these same quantities. The task is then to solve the appropriate system of equations. For instance in the case of $n = 3$ we have

$$E \det M(x) = \det M,$$

$$E(\det M(x))^2 = (\det M)^2 + \rho \Sigma(M^{ij})^2 + 2\rho^2 \Sigma M^2_{ij} + 6\rho^3,$$

$$E(\det M(x))^3 = (\det M)^3 + 3\rho(\det M)\Sigma(M^{ij})^2,$$

$$+ 12\rho^2(\det M)\Sigma M^2_{ij} + 60\rho^3 \det M.$$

Thus—at least if $\det M \neq 0$—we can solve for $\Sigma(M^{ij})^2$ and $\Sigma M^2_{ij}$ as long as $\rho \neq 0$. The case of $\det M = 0$ requires the computation of the fourth moment.

In general one ends up with the determinant of $M$, and the sum of squares of all its $(n - k) \times (n - k)$ minors for each $k = 1, \cdots, n - 1$. It is now a simple exercise using the Cauchy-Binet formula to see that these quantities give exactly the eigenvalues of $MM^*$. A complete proof of Theorem II, as well as a discussion for the case of a general correlation matrix $R$, will appear elsewhere.

REFERENCE


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