I. Introduction. Oberst [7], Laudal [4], Watts [10], and André [1] have shown that derived functors of colimit define a homology theory for \text{Cat}, the category of small categories. In this note, we outline a proof of uniqueness for such a homology theory, making extensive use of a Kan-type construction (see e.g. Lemma A) and of uniqueness for homology in \text{S}^{\Delta \text{op}} \text{P}, the category of simplicial sets [2].

II. Preliminaries. The following Kan-type construction is used in several contexts.

**Lemma A.** Let \( \mathcal{C} \) be a cocomplete category, \( \mathcal{C} \) a small category, and \( \theta: \mathcal{C} \rightarrow \mathcal{C} \) a functor. Then there exists an adjoint pair: the singular functor \( S_{\theta}: \mathcal{C} \rightarrow \mathcal{S}^{\text{C} \text{op}} \) defined by \( S_{\theta}(A) = \mathcal{C}(\theta \_ , A) \), for \( A \in |\mathcal{C}| \), and its left adjoint \( \hat{\theta}: \mathcal{S}^{\text{C} \text{op}} \rightarrow \mathcal{C} \).

Let \( \Delta \) be the small category whose objects are the finite ordinals \( [k] = \{0 < 1 < 2 < \ldots < k\} \) and whose morphisms are order preserving functions \( \mu: [k] \rightarrow [m] \). By considering the full inclusion functor \( \iota: \Delta \rightarrow \text{Cat} \), in the context of Lemma A, nerve, \( N: \text{Cat} \rightarrow \mathcal{S}^{\Delta \text{op}} \), is the singular adjoint of categorical realization \( c: \mathcal{S}^{\Delta \text{op}} \rightarrow \text{Cat} \) and \( cN = \text{id}_{\text{Cat}} \) [3, p. 33]. Thus the standard representable \( k \)-dimensional simplicial set \( \Delta[k] \) is actually \( N([k]) = \Delta(-, [k]) \).

Similarly, the functor \( r: \Delta \rightarrow \text{Cat} \) defined as the comma category, \( r[k] = \Delta \downarrow [k] \), gives rise to another pair of adjoint functors \( S: \text{Cat} \rightarrow \mathcal{S}^{\Delta \text{op}} \) and \( \Gamma: \mathcal{S}^{\Delta \text{op}} \rightarrow \text{Cat} \). Let \( X \in |\mathcal{S}^{\Delta \text{op}}| \), then \( \Gamma X \) is the small category whose objects are \( \Pi_{k \geq 0} X_k \), and whose morphisms are triples \( (y, \mu, x) \) where \( x \in X_m \) is the codomain, \( \mu: [k] \rightarrow [m] \) in \( \Delta \) is the morphism, and \( y = X(\mu)x \) in \( X_k \) is the domain.

The natural transformation "last", \( \eta: r \rightarrow \iota \), is given by \( \eta_k(\alpha: [p] \rightarrow [k]) = \alpha(p) \in [k] \). By adjoint functor theory and by the theory of coends, \( \eta \) induces natural transformations \( \eta^1: N \rightarrow S \), \( \eta^2: \Gamma \rightarrow c \), \( \eta^3: \Gamma N \rightarrow cNX = \text{id}_{\text{Cat}} \), and \( \eta^4: \Gamma N \rightarrow \text{id}_{\mathcal{S}^{\Delta \text{op}}} \).
The Milnor geometric realization functor \(|\cdot|: S^\Delta^{op} \rightarrow \text{Top}\) [5], \(\text{Top}\) the category of CW complexes, can also be viewed as another example of the "Lemma A" situation. If \(\theta: \Delta \rightarrow \text{Top}\) is given by \(\theta([k]) = \Delta^k\), the standard \(k\)-dimensional affine simplex in \(\mathbb{R}^{k+1}\), then \(|\cdot|: S^\Delta^{op} \rightarrow \text{Top}\) is the left adjoint of the singular complex functor \(S_c: \text{Top} \rightarrow S^\Delta^{op}\). Let \(B = |N_\cdot|: \text{Cat} \rightarrow \text{Top}\) denote the Segal classifying space functor [9]. Then for each small category \(C\), \(BC\) is the CW complex whose \(k\)-cells are in one-to-one correspondence with nondegenerate \(k\)-simplicies of \(NC\).

### III. Definition of homology and existence.
A subcategory \(C'\) of \(C\) is initial in \(C\) if all morphisms \(m: p \rightarrow q\) in \(C\) with codomain in \(C'\) are in \(C'\). A pair of small categories \((C, C')\) is said to be admissible if \(C'\) is an initial subcategory of \(C\). The category of all admissible pairs and obvious morphisms is also denoted by \(\text{Cat}\).

A homology theory for \(\text{Cat}\) is a pair \((h, \partial)\), where \(h: \text{Cat} \rightarrow \text{Ab}^Z\) is a functor from the category of admissible pairs to graded abelian groups and \(\partial_*: h_* \rightarrow h_{*-1}\) is a natural transformation of degree \(-1\), satisfying the standard Eilenberg-Steenrod-Milnor axioms [6]: dimension, exactness, excision, homotopy and strong additivity. We state the last three below.

**Excision Axiom.** Let \(C\) be any small category with initial subcategories \(C_1\) and \(C_2\). Then the inclusions induce \(h_*(C_2, C_1 \cap C_2) \cong h_*(C_1 \cup C_2, C_1)\), where \(C_1 \cap C_2\) and \(C_1 \cup C_2\) are subcategories of \(C\) making the following square bicartesian in \(\text{Cat}\):

\[
\begin{array}{ccc}
C_1 \cap C_2 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_2 & \rightarrow & C_1 \cup C_2 \\
\end{array}
\]

**Homotopy Axiom.** If \(F: C \rightarrow D\) is a weak homotopy equivalence, i.e. if \(BF: BC \rightarrow BD\) is a homotopy equivalence in \(\text{Top}\), then \(h_*F: h_*C \rightarrow h_*D\) is an isomorphism.

**Strong Additivity (Milnor) Axiom.** Let \(\{(C_\alpha, C'_\alpha) | \alpha \in A\}\) be a collection of admissible pairs in \(\text{Cat}\). Then the inclusions induce

\[
\bigoplus_\alpha h_*(C_\alpha, C'_\alpha) \cong h_*\left(\bigsqcup_\alpha C_\alpha, \bigsqcup_\alpha C'_\alpha\right).
\]

We assume that the coefficient group \(A \in \text{Ab}\) is fixed. More general coefficient systems will be discussed in the longer exposition.

Define \(\Delta_{C,C}(A): C \rightarrow \text{Ab}\) by
on objects and in the obvious fashion on morphisms.

**Remark.** \( C' \) initial in \( C \) guarantees that \( \Delta_{C,C'}(A) \) is a functor. Other pairs, e.g., \( C' \) terminal in \( C \) or \( C' \) an "interval" in \( C \) would also satisfy this condition.

**Theorem 1.** \( \langle H, \partial \rangle \) is a homology theory for \( \text{Cat} \), where

\[
H_*(C, C') = L_* \text{colim}_C \Delta_{C,C'}(A),
\]

\( L_* \text{colim}_C \colon \text{Ab}^C \to \text{Ab} \) being the left derived functors of \( \text{colim}_C \colon \text{Ab}^C \to \text{Ab} \).

**Proof.** See [7] and [4].

Using the canonical coflabby resolution of \( \Delta_{C,C'}(A) \) ([10], [4], [7]) we see that the complexes used to calculate the homology yield the following:

**Corollary.** Let \( \langle H, \partial \rangle \) be the unique homology in \( S^A^{\text{op}} \) [2], i.e. singular homology. Then

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{N} & S^A^{\text{op}} \\
H_* & \downarrow & H_* \\
\text{Ab} & \xrightarrow{Z} & \text{Ab} \\
\end{array}
\]

commutes up to isomorphism.

**IV. Uniqueness.** The proof of our uniqueness theorem rests on the following two lemmas both of which are used in applying the homotopy axiom.

**Lemma B.** The natural transformation \( \eta^4 \colon NT \to \text{id}_{S^A^{\text{op}}} \) induces a natural homotopy equivalence \( \eta^4_X \colon NTX \to |X| \) in \( \text{Top} \), for every simplicial set \( X \).

**Lemma C.** The natural transformation \( \eta^3 \colon \Gamma N \to \text{id}_{\text{Cat}} \) is a weak homotopy equivalence in \( \text{Cat} \), i.e. \( \eta^3 \colon \Gamma NC \to C \) is a weak homotopy equivalence for each small category \( C \).

**Theorem 2.** If \( \langle h, \partial \rangle \) is a homology theory for \( \text{Cat} \), then

\[
\begin{array}{ccc}
S^A^{\text{op}} & \xrightarrow{\Gamma} & \text{Cat} \\
H_* & \downarrow & H_* \\
\text{Ab} & \xrightarrow{Z} & \text{Ab} \\
\end{array}
\]

commutes up to isomorphism.
The proof consists of showing that $h_\# \Gamma: S^{\Delta \text{op}} \rightarrow \text{Ab}^Z$ satisfies the standard axioms for homology theory in $S^{\Delta \text{op}}$. Hence, by uniqueness of such a theory [2], the diagram commutes. Some of the special properties of $\Gamma: S^{\Delta \text{op}} \rightarrow \text{Cat}$ used in the proof are that $\Gamma$ commutes with pullbacks, $\Gamma(\Delta [k])$ is contractible, and $(\Gamma X, \Gamma X') = \Gamma(X, X')$ is an admissible pair. Lemma B is needed in proving the homotopy axiom.

**THEOREM 3 (UNIQUENESS).** If $(h, \partial)$ is a homology theory for $\text{Cat}$ then $h_\#(C, C') \cong H_\#(C, C')$.

**PROOF.** $h_\#(C, C') \cong h_\#(\Gamma NC, \Gamma NC')$ by the homotopy axiom used in conjunction with Lemma C. By Theorem 2, $h_\#(\Gamma NC, \Gamma NC') \cong H_\#(NC, NC')$. But the Corollary guarantees that $H_\#(NC, NC') \cong H_\#(C, C')$. Q.E.D.

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**BIBLIOGRAPHY**