A RESTRICTION THEOREM FOR THE FOURIER TRANSFORM

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Let \( f \) be a Schwartz function on \( \mathbb{R}^n \), and let \( \hat{f}(\theta) \) denote the restriction of the Fourier transform of \( f \) to the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). We prove

**Theorem.** If \( f \) is in \( L^p(\mathbb{R}^n) \) for some \( p \) with \( 1 < p < 2(n + 1)/(n + 3) \), then

\[
\int_{S^{n-1}} |\hat{f}(\theta)|^2 \, d\theta \leq c_p \|f\|_p^2.
\]

**Proof.**

\[
\int |\hat{f}(\theta)|^2 \, d\theta = \int f * \overline{f}(x) \, d\theta(x) = \int f(x) \, \overline{\hat{f}(x)} \, dx \leq \|f\|_p \|\overline{\hat{f}} \ast f\|_p,
\]

for conjugate indices \( p \) and \( p' \). Thus it suffices to prove that the operator given by convolution with \( \overline{\hat{f}} \) is bounded from \( L^p \) to \( L^{p'} \) for \( p \) in the appropriate range. Let \( K(x) \) be a radial Schwartz function with \( K(x) = 1 \) for \( |x| < 100 \), and let \( T_k(x) = (K(x/2^k) - K(x/2^{k-1})) \, \overline{\hat{f}(x)} \). It suffices to show there exists \( \varepsilon = \varepsilon(p) > 0 \) such that \( \|T_k \ast f\|_{p'} \leq C 2^{-\varepsilon k} \|f\|_p \). This follows from interpolating the estimates \( \|T_k \ast f\|_{\infty} \leq C 2^{-(n-1)k/2} \|f\|_1 \) and \( \|T_k \ast f\|_2 \leq 2^k \|f\|_2 \).

Professor E. M. Stein has extended the range of this result to include \( p = 2(n + 1)/(n + 3) \). His proof uses complex interpolation of the operators given by convolution with the functions \( B_\alpha(x) = J_\alpha(2\pi|x|)/|x|^\alpha \). Then \( \hat{\theta}(x) = B_{(n-2)/2}(x) \).

A great deal was previously known about such restriction theorems. E. M. Stein originally established the theorem for \( 1 \leq p < 4n/(3n + 1) \). For \( n = 2 \), this was extended by Fefferman and Stein [2] to the range \( 1 \leq p < 6/5 \). P. Sjolin (see [1]) proved the theorem for \( n = 3 \) and \( 1 \leq p < 4/3 \). Finally, A. Zygmund [3] determined for two dimensions all \( p \) and \( q \) such that the Fourier transform of an \( L^p \) function restricts to \( L^q(S^1) \). Since a

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very simple homogeneity argument shows that the theorem fails for
\( p > 2(n + 1)/(n + 3) \), the present result, together with the result of Stein,
represent the optimal estimate of this sort.

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