We announce several theorems which suggest a minimal classification of relative equilibria in the planar $n$-body problem. These theorems also answer several questions on the nature of degenerate relative equilibria classes which were asked recently by S. Smale [3]. A summary of previous results can be found in an earlier paper [1]. It is a pleasure to thank S. Smale for encouragement in this work.

1. Morse theory and relative equilibria. We study the critical set of a real analytic function $V_m < 0$ on a real analytic manifold $X_m$ where $n > 3$ and $m = (m_1, \ldots, m_n) \in \mathbb{R}_+^n$ are fixed. Critical points of $V_m$ correspond in a 1-1 fashion to classes of relative equilibria. $V_m$ always has a compact critical set which we may investigate by Morse theory even when degenerate critical points exist [2].

The integral singular homology of $X_m$ (a manifold which is homeomorphic to a Stein manifold $P_{n-2}(\mathbb{C}) - \Delta_{n-2}$) is given by a recurrence relation [1]. This suggests that there is a uniform lower bound on the number of critical points of each index of $V_m$ which is given by recurrence. As a first step toward classifying relative equilibria Theorem 1 gives such a relation.

In Theorem 2 we assert that $V_m$ is a Morse function for any $n > 3$ and for almost all $m \in \mathbb{R}_+^n$ (in the sense of Lebesgue measure).

Theorem 3 answers the question: Is $V_m$ always a Morse function?

Finally, we examine the case of four masses to show how a degeneracy of $V_m$ arises. An interpretation of Theorem 1 in the degenerate case sheds light on the creation and annihilation of relative equilibria.

2. Main theorems. In this paragraph for any $i$, $0 < i < 2n - 4$, let $\mu_i(n)$ denote a uniform lower bound to the number of critical points of $V_m$.
with index equal to $2n - 4 - i$ whenever $\mathring{V}_m$ is a Morse function. By [1, Theorem 2] the index of any critical point of $\mathring{V}_m$ whether degenerate or not has $n - 2$ as a lower bound.

**Theorem 1.** For any $n \geq 3$ and for any $i$, $0 \leq i \leq n - 2$, $\mu_i(n) = (n - 1 - i) \mu_i(n - 1) + (2n - 2 - i) \mu_{i-1}(n - 1)$; and $\mu_i(n) = 0$ for $i > n - 2$.

**Corollary 1.1.** $\mu_i(n) = C_{n,i}(n - 1 - i) (n - 2)!$ for any $i$, $0 \leq i \leq n - 2$, and for any $n \geq 3$.

Here $C_{n,i}$ is the binomial coefficient.

**Corollary 1.2.** $\Sigma_{i=0}^{n-2} \mu_i(n) = [2^{n-1}(n - 2) + 1] (n - 2)!$ for any $n \geq 3$.

Let $\beta_i = \text{rank } H_i(P_{n-2}(C) - \delta_{n-2})$ for any $i$, $0 \leq i \leq 2n - 4$, and $n \geq 3$ where $H_*$ is the integral singular homology. We write $A(t) > B(t)$ for any two polynomials $A(t), B(t)$ provided that $A(t) - B(t) = (1 + t) C(t)$ where $C(t)$ has nonnegative coefficients. This relation subsumes the Morse inequalities.

**Corollary 1.3.** $\Sigma_{i=0}^{n-2} \mu_i(n) t^i > \Sigma_{i=0}^{n-2} \beta_i t^i$ for any $n \geq 3$.

Recently, S. Smale [3] has raised questions about the nature of the set of masses $\Sigma_n \subset R^n_+$ on which degeneracies of $\mathring{V}_m$ arise. The next two theorems give some measure-theoretic properties of $\Sigma_n$.

**Theorem 2.** $\mathring{V}_m$ is a Morse function for any $n \geq 3$ and for almost all masses $m \in R^n_+$ (in the sense of Lebesgue measure).

**Corollary 2.1.** There are only finitely many relative equilibria classes in the planar $n$-body problem for any $n \geq 3$ and for almost all masses $m \in R^n_+$.

**Remark.** It is an open question whether for some $n \geq 4$ and $m \in R^n_+$ there are infinitely many critical points of $\mathring{V}_m$.

Theorem 2 shows that $\Sigma_n$ has measure 0 for all $n \geq 3$. By [1, Theorem 4] we have $\Sigma_3 = \emptyset$. The next result shows that for $n \geq 4$ degeneracies arise.

**Theorem 3.** $\Sigma_n \neq \emptyset$ for any $n \geq 4$.

3. **Classifying relative equilibria.** For any three positive masses there are precisely five critical points of $\mathring{V}_m$ and these critical points are nondegenerate. Their distribution corresponds to that of the minimal classification given by Theorem 1.

For $n = 4$ masses a degeneracy arises in the following fashion. In the
plane $E^2$ we place three unit masses at the vertices of an equilateral triangle with center of mass at the origin. We place at the origin an arbitrary fourth positive mass, $m_4$. It follows easily for all values of $m_4$ that this configuration is a relative equilibrium.

Let $m = (1, 1, 1, m_4)$ and let $x \in X_m$ be the relative equilibria class to which the above relative equilibrium belongs. Let $D^2 \tilde{V}_m(x)$, the hessian of $\tilde{V}_m$ at $x$, a real symmetric bilinear form on $T_x X_m$, be considered a function of $m_4$. By direct calculation [2] we find that the hessian is degenerate if and only if $m_4$ equals the unique positive number $m^*_4$ which is given by $m^*_4 = (2 + 3\sqrt{3})/(18 - 5\sqrt{3}) < 1$.

For $m_4 < m^*_4$ the index of $x$ (i.e. the index of the hessian of $\tilde{V}_m$ at $x$) equals 4 and $x$ is a nondegenerate local maximum of $\tilde{V}_m$. For $m_4 \geq m^*_4$ the index of $x$ equals 2. When $m_4 = m^*_4$ the dimension of the nullspace of the hessian equals 2. This is the maximum degeneracy possible for four masses.

These considerations suggest the following interpretation of Theorem 1 whenever $\tilde{V}_m$ has isolated degenerate critical points.

For any $n \geq 4$ let $m \in R^m_n$ be such that $\tilde{V}_m$ has only isolated critical points. Let $c_1 < \ldots < c_r < 0$ be the critical values of $\tilde{V}_m$. Set $c_0 = -\infty$ and for any $j$, $1 \leq j \leq r$, define $W_j = \tilde{V}_m^{-1}(c_{j-1}, c_j)$. Let $\Lambda_j$ be the set of critical points at level $j$, $1 \leq j \leq r$. Finally, for any $i$, $0 \leq i \leq 2n - 4$, define $\tau_i(n, m)$ by

$$\tau_i(n, m) = \sum_{j=1}^r \text{rank} H_{2n-4-i}(W_j \cup \Lambda_j, W_j).$$

By [1, Theorem 2] we have $\tau_i(n, m) = 0$ for any $i > n - 2$.

**Theorem 4.** For any $n \geq 4$ and any $m \in R^m_n$ for which $\tilde{V}_m$ has only isolated critical points, $\tau_i(n, m) \geq \mu_i(n)$ for any $i$, $0 \leq i \leq 2n - 4$.

**Corollary 4.1.** $\sum_{i=0}^{n-2} \tau_i(n, m) t^i > \sum_{i=0}^{n-2} \mu_i(n) t^i$ for any $n \geq 4$.

**REFERENCES**

