MEASURES AS CONVOLUTION OPERATORS
ON HARDY AND LIPSCHITZ SPACES

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In this note we announce some new results concerning the spectral theory of measures as convolution operators. To state our principal theorem, we introduce the following notation. If $X$ is a Banach space and $T$ is a bounded linear operator on $X$, we write $\text{sp}(T, X)$ to denote the spectrum of $T$ on $X$. Let $G$ be an LCA group with dual group $\Gamma$. $M(G)$ will denote the class of finite regular Borel measures on $G$, and $M_0(G) = \{ \mu \in M(G) | \hat{\mu} \text{ vanishes at infinity on } \Gamma \}$. For $\mu \in M(G)$, let $T_\mu$ denote the operator defined by $T_\mu(f) = \mu * f$, that is, convolutions with $\mu$. Finally, let $H^1$ be the natural domain of the Hilbert transform on $L_1(\mathbb{R})$, and let Lip $\alpha$ denote the usual class of bounded functions on $\mathbb{R}$ satisfying a Lipschitz condition of order $\alpha$, $0 < \alpha < 1$. We can now state our main result.

**Theorem 1.** There exists a measure $\mu \in M_0(\mathbb{R})$ such that

(a) $\text{sp}(T_\mu, H^1) \neq \hat{\mu}(\mathbb{R}) \cup \{0\}$, and

(b) $\text{sp}(T_\mu, \text{Lip } \alpha) \neq \hat{\mu}(\mathbb{R}) \cup \{0\}$, $0 < \alpha < 1$.

This may be viewed as an analogue of the now classical Wiener-Pitt theorem concerning the invertibility of Fourier-Stieltjes transforms [4, Theorem 5.3.4]. Moreover, an elementary interpolation argument shows that if $1 < p < \infty$,

$$\text{sp}(T_\nu, L_p) = \hat{\nu}(\mathbb{R}) \cup \{0\},$$

for all $\nu \in M_0(\mathbb{R})$ (see [1, §1.4]). Thus, in a sense, our theorem is intermediate between the $L_1$ and $L_p$ ($1 < p < \infty$) cases.

The proof of Theorem 1 is based on the following result.

**Theorem 2.** Let


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\[
\psi_{\delta}(x) = \begin{cases} 
1/\delta & \text{if } 0 \leq x < \delta/2, \\
-1/\delta & \text{if } -\delta/2 < x < 0.
\end{cases}
\]

Then there exists a measure \( \nu \in M_0(\mathbb{R}) \), of total variation norm 1, which satisfies

\[
(\ast) \quad \limsup_{\delta \to 0} \| \nu^j \ast \psi_{\delta} \|_1 = 1,
\]

for all \( j = 1, 2, \cdots \).

The expression \((\ast)\) arises since, as is readily verified,

\[
\limsup_{\delta \to 0} \| T(\psi_{\delta}) \|_1 \leq C \| T \|_{O(H^1)}
\]

for every bounded linear operator \( T \) on \( H^1 \). Here \( C \) is an absolute constant and \( \| T \|_{O(H^1)} \) denotes the operator norm of \( T \) on \( H^1 \). Moreover, if \( g \in L_1(\mathbb{R}) \), \( \lim_{\delta \to 0} \| g \ast \psi_{\delta} \|_1 = 0 \). Therefore, the expression \((\ast)\) provides us with a lower bound for the norms \( \| (T_{\nu} - T_g)^j \|_{O(H^1)} \), \( j = 1, 2, \ldots \), for every \( g \in L_1(\mathbb{R}) \). Consequently, since \( \hat{\nu} \) vanishes at infinity, we have for appropriate \( f \in L_1(\mathbb{R}) \) that \( \| \nu - f \|_{L_\infty(\mathbb{R})} < 1 \), whereas the spectral radius of the operator \( T_{\nu - f} \) on \( H^1 \) is at least 1.

A similar argument also applies to the space \( \text{Lip} \alpha \) and certain of its variants, specifically, certain of the Taibleson spaces (see [5]). Thus Theorem 1 follows from Theorem 2 (with \( \mu = \nu - f \)).

The objects of study on Theorem 2 are measures of Cantor-Lebesgue type, which are subject to certain arithmetic constraints. Specifically, we examine infinite Bernoulli convolutions of the form

\[
\nu = \ast_{k=1}^{\infty} (\frac{1}{2} \delta_0 + \frac{1}{2} \delta_k t_k),
\]

where the positive sequence \( \{t_k\} \in l_1 \) is chosen so that

1. \( t_{k+1}/t_k \to 0 \) as \( k \to \infty \), and \( t_n > \Sigma_{k=n+1}^{\infty} t_k, n = 1, 2, \ldots \),
2. \( \hat{\nu} \) vanishes at infinity on \( \mathbb{R} \), and
3. \( \{t_k\} \) is fully independent, that is, if \( \{n_k\} \) is any bounded sequence of integers, and if \( \Sigma_{k=1}^{\infty} n_k t_k = 0 \), then \( n_k = 0, k = 1, 2, \ldots \).

The existence of such sequences is guaranteed by probabilistic considerations (see [3, pp. 256–258]).

The proof of Theorem 2 then consists largely of a careful study of the \( j \)-fold sum of the Cantor set \( \{ \Sigma_{k=1}^{\infty} \epsilon_k t_k \mid \epsilon_k = 0 \text{ or } 1 \} \) generated by sequences.
\{t_k\} of the above form. In particular, we show that the \(j\)-fold sum itself “looks like” a Cantor-type set which has been constructed in a “regular” way. We then integrate along the gaps arising at the various stages of the construction, to obtain estimate (*) in Theorem 2.

Finally, we remark that the techniques used here also yield the analogue of Theorem 1 for the circle group. Further results, detailed proofs, and some applications of this theory will appear in [6].

REFERENCES

6. M. Zafran, *Measures as convolution operators on \(H^1\) and Lip \(\alpha\) (submitted).*

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