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Map color theorem, by Gerhard Ringel, Springer-Verlag, New York, Heidelberg, Berlin, 1974, 191+xii pp., $22.20

The four color conjecture is a famous problem that has challenged and stimulated mathematicians for more than a century. As most mathematicians know, it consists of the statement that with four colors one can color any map on a sphere such that any two countries with a boundary edge in common are of different colors. The present volume concerns a related problem: how many colors are necessary to color all similarly colored maps on surfaces of higher genus?

This problem has an entirely different flavor, as we shall see; it has a long history as well. It was posed by Heawood, who thought he had proven his conjectured answer in 1890. The last case was solved in 1968 (most cases solved by the author) verifying the original conjecture. A complete description in remarkably clear language of solutions for all cases is presented in this volume, which is written at a level suitable for an undergraduate seminar.

The major difference between the sphere problem and higher genus-surface problems is this: On a sphere one knows that one cannot have five countries every pair of which are neighbors—a configuration obviously requiring five colors—but one does not know if there is some large configuration of countries that somehow requires five colors. (We now know that upward of 40 countries would be required in such a configuration.) On every other locally planar surface, on the other hand, the interesting question is: How many countries can one have such that each is the neighbor of each other? It is easy to show from Euler's formula that the largest chromatic number or number of colors needed for any graph embeddable on any such surface is the size of the largest "complete graph" that can be drawn on it without boundary crossings. Thus, the "Heawood mapping problem" considered here can be posed as the question: What is the smallest genus surface into which the complete graph on n vertices can be embedded? A lower bound on this number for each n follows from Euler's formula. The remaining problem is therefore constructive: Can one embed the complete graph on n vertices without crossing in the smallest genus surface allowed by Euler's formula? This is a problem of combinatorial topology. The answer is yes in every case.

To obtain this answer it is necessary to describe how one characterizes an embedding, and how one can go about finding embeddings of complete graphs. For certain n values the embedding sought is a "triangulation" of the surface involved and can be characterized by listing the triangles in it, or more concisely by listing the cycle of vertices in clockwise or counterclockwise order as seen from each vertex. If the lists yield consistent triangles (each vertex and each consecutive pair on its cycle form a triangle) they characterize a triangulation and hence an embedding. In the other cases one can seek triangulations of the complete graph less some edges in surfaces of lower genus, and add the missing edges with the extra "handles" supplied by the missing genus.

The problem can therefore be reduced to finding triangulations. A number of ingenious methods have been developed for obtaining these—somewhat more than half of the book is devoted to obtaining the desired triangulations for the various cases. It is convenient to look for cycles of vertices seen in order from one vertex such that the cycle for the kth vertex can be obtained from that from the 0th by adding k mod n to the index of every vertex in the cycle. (There are other simple analogous possibilities that can also be sought for the interrelationships of these cycles.) The conditions under which this can be done for a given n are equivalent to the cycles being obtainable from an assignment of integer (more generally, group-element) flows to the arcs of a certain graph so as to conserve flow at each vertex. The flow conservation represents the consistency of the triangles following from the cycles.

In this volume schemes are developed for constructing the needed flow assignment for the various infinite classes of situations encountered. A few cases require special treatment, the solutions being expressed as lists of cycles. Solutions of index 2 and 3 (that have 2 or 3 different cycles from which all the others can be obtained by addition) are also obtained.

The exposition is clear throughout. There is a slight tendency for the results to be presented in a fashion that makes them seem dramatic. Thus the Euler-formula bound obtained by Heawood is expressed as a mysterious looking formula and some of the reductions seem to come almost by magic. This has some pedagogic value, but perhaps less than an approach that relies on explanation to motivate the results.

There is also a slight tendency to extract motivation for the work by its relation to the four color problem. That relation is, however, only that the problems share some language. Euler's formula implies that the hard part of the four color problem is lacking for analogous problems involving five or more colors. (For example we do not know how to prove that the minimal number of vertices of a planar graph requiring five colors is not both finite and above one billion.)

One suspects that the author was not certain that the subject matter in itself, the minimal genus embedding of complete graphs, was of sufficiently wide interest to attract and keep his audience. If so, he should not have worried. The problem is of interest in itself; study of the material in this book can deepen a student's comprehension of topological concepts, and give him exposure to clear incisive elementary combinatorial arguments. Not everyone will want to pursue every case to its denouement—but even this is fine in a seminar; each student can choose one set of cases, pursue it on his own and construct a report on it.

Study of this material would in my opinion make for a very nice undergraduate seminar for students with a variety of backgrounds.

The critical flaw in this work, from the viewpoint of classical mathematics, is that, while the problem is clearly defined, the methodology is rather special, introduced ad hoc and without developed structure. Here one seeks embeddings of various graphs, and finds them, some by special techniques, others by somewhat more general ones. For each case one pulls a trick out of a bag. In short, the result is here presented as a piece of applied mathematics, albeit one originating from a pure mathematical context. The thrust of the work is the presentation of some techniques that succeed for each of the cases of interest in this problem. (As a matter of fact there are related practical applied problems that arise in the construction of integrated circuits.)

From a pure mathematical standpoint a number of questions are left open. Having seen the tricks one longs to analyze the bag. Some of these may be worthwhile research topics:

1. The methods used here for constructing flow patterns in appropriate classes of graphs with appropriate groups deserve study in themselves.

Can they be extended to wider contexts than the cases arising in this problem—other graphs, groups; what is their ultimate power?

2. How far can one go toward embedding other classes of graphs or toward results about an arbitrary graph on n vertices? Can one obtain more detailed results such as statements about the existence of embeddings with given index?

3. How do the structures produced here—lists of cycles or flow graphs or whatever—relate to other combinatorial structures—block designs, Latin squares, etc?—is there any relation that allows nontrivial implication in either direction—from or to these results?

4. Is there anything at all in this work that is relevant to integrated circuit problems?

In a similar vein to the given problem are some more difficult questions. The results presented here relate to embeddings of embeddable graphs—when a graph is not embeddable one can raise an analogous question: How much crossing of edges is required to embed the graph? This problem seems more difficult than the original one, because one lacks a characterization of the nature of the "best" embedding; one cannot look for a solution as a triangulation. Very little is known about such "crossing number" problems except conjectured upper bounds on crossing number based upon obvious constructions. Their solution probably awaits a new set of ideas.

Thus, although this volume contains an extremely lucid presentation of a complete solution to the Heawood mapping problem, it should not necessarily be considered the last word on the general subject. Of course this is only one more reason why it deserves to be read.

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Differential analysis on complex manifolds, by R. O. Wells, Jr., Prentice-Hall, Englewood Cliffs, N.J., 1973, x+252 pp., $13.95.

Until the late 1940's it seems that compact complex manifolds were only studied occasionally and even then were not studied as a class of intrinsically interesting objects. In fact most examples of compact complex manifolds were either submanifolds of complex projective space P^n (such manifolds will be called *projective*) or else were Kähler manifolds (such as the nonalgebraic tori). It seems that Hopf's simple construction in 1948 of a non-Kählerian compact complex manifold with $\mathbf{C}^2-\{(0, 0)\}$ as universal covering made the study of complex manifolds much more interesting.