LOOP SPACES AND FINITE ORTHOGONAL GROUPS

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Communicated by Edgar H. Brown, Jr., February 25, 1975

In this note we announce the results of our computations of the mod 2 homology of the orthogonal groups $O(n, \mathbb{F}_q)$ over finite fields $\mathbb{F}_q$ of characteristic $p \neq 2$. We have obtained a 2-local equivalence between the infinite loop space associated with these orthogonal groups and the homotopy fiber $JO(q)$ of the map $(\psi^q - 1): BO(\mathbb{R}) \to BSO(\mathbb{R})$, where $\psi^q$ is the Adams operation. For $q \equiv \pm 3 \pmod{8}$, these spaces $JO(q)$ are of considerable geometric interest, since $n^*JO(q)$ is essentially the image of $J*: \pi_*SO(\mathbb{R}) \to \pi_*\mathcal{S}F$ at the prime 2. Here $J: SO(\mathbb{R}) \to \mathcal{S}F$ is the $J$-homomorphism of G. Whitehead.

Since the Whitney sum induces an infinite loop space structure on $JO(q)$, we can define Dyer-Lashof operations on its homology. We have computed $H_*(JO(q), \mathbb{Z}_2)$ as an algebra over the Dyer-Lashof algebra.

Our main results are as follows:

**Theorem 1.** There is an equivalence of infinite loop spaces

$$\Gamma_0BO(\mathbb{F}_q) \cong_{(2)} JO(q)_{(2)}.$$ 

Here $\Gamma_0BO(\mathbb{F}_q)$ denotes the 0-component of the group completion of $\Pi_{n=0}^{\infty}BO(n, \mathbb{F}_q)$. See May [3] for details.


1This paper is partially based on the first author's Ph. D. thesis prepared at the University of Chicago under the direction of J. Peter May.
2Partially supported by NSF graduate Fellowship.
3Partially supported by NSF grant GP25335 and a Science Research Council of Britain Fellowship.
We should point out that in a recent paper [1], E. M. Friedlander, using the methods of etale homotopy theory, has established that there is an equivalence of spaces \( \Gamma_0 BO(F_q) \xrightarrow{\sim} JO(q) \). However there seems to be no apparent way of showing that his map is an infinite loop map.

Using the equivalence of Theorem 1, we can now compute homology operations in \( H_\ast(JO(q), \mathbb{Z}_2) \) by the methods employed in Priddy [4]. We obtain the following result:

**Theorem 2.** \( H_\ast(JO(q), \mathbb{Z}_2) = \mathbb{Z}_2[\nu, \nu_2, \ldots] \otimes E \{ \bar{u}_i | i \geq 1 \}, \) where the \( \bar{u}_i \)'s are explicitly chosen classes which map onto the standard generators \( \{ \bar{c} \} \) of \( H_\ast(BO(R), \mathbb{Z}_2) \) and the \( \bar{u}_i \)'s come from the standard generators \( u_i \) of \( H_\ast(SO(R), \mathbb{Z}_2) \). The action of the Dyer-Lashof algebra on \( H_\ast(JO(q), \mathbb{Z}_2) \) is given by

1. \( Q^n(\bar{u}_k) = \sum_{a+b+c=n+k} (k-a, n-k-b-1) \bar{u}_a \cdot \bar{u}_b \cdot \bar{u}_c; \)

2. \( Q^n(\bar{u}_k \ast [1]) = \sum_{a+b+c+d=n+k} (n-k-1, a+c-n) \bar{u}_a \cdot \bar{u}_b \cdot \bar{u}_c \cdot \bar{u}_d \) \( \ast [2] \)

if \( q \equiv \pm 3 \pmod{8} \);

3. \( Q^n(\bar{u}_k \ast [1]) = \sum_{a+b=n+k} (n-k-1, a-n) \bar{u}_a \cdot \bar{u}_b \) \( \ast [2] \)

if \( q \equiv \pm 1 \pmod{8} \).

The generators \( \{ \bar{u}_i \} \) of Theorem 2 can be chosen in two different ways corresponding to two different lifts into \( JO(q) \) of the inclusion \( RP^\infty = BO(1, R) \rightarrow BO(R) \). There is an involution \( \Phi: JO(q)_{(2)} \rightarrow JO(q)_{(2)} \) of infinite loop spaces which sends one lift to the other. In \( \Gamma_0 BO(F_q) \) this involution is induced by conjugation by the matrix \( \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) where \( a, b \) are elements of \( F_q \) such that \( a^2 + b^2 \) is not a square.

We remark here that our computations produce a much simpler derivation of the formula for the Dyer-Lashof operations on \( H_\ast(SO(R), \mathbb{Z}_2) \) than the one in Kochman [2].

To obtain generators for the homology groups of \( O(n, F_q) \), we proceed as follows. Since \( O(1, F_q) = \mathbb{Z}_2, H_k(BO(1, F_q), \mathbb{Z}_2) = \mathbb{Z}_2 \), and we denote by \( \nu_k \) the generator of this group. Let \( T \subset O(2, F_q) \) be generated by \( \{ B, -B \} \) where \( B = \Phi(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \). Then \( T = \mathbb{Z}_2 \times \mathbb{Z}_2 \), and we denote by \( \nu_{k,l} \) the image of \( x_k \otimes x_l \) under

\[
H_k(RP^\infty, \mathbb{Z}_2) \otimes H_l(RP^\infty, \mathbb{Z}_2) \rightarrow H_{k+l}(BT, \mathbb{Z}_2) \\
\rightarrow H_{k+l}(BO(2, F_1), \mathbb{Z}_2).
\]
Then we have

**Theorem 3.** As an algebra, $\bigoplus_{n=0}^{\infty} H_*(BO(n, F_q), \mathbb{Z}_2)$ is commutative with generators $v_i, v_{i,j}$, $i, j \geq 0$, subject to the relations

1. $v_{i,j} = v_{j,i}$,
2. $v_{i,i} = v_i^2$,
3. $v_{i,j}^2 = v_{i,k}v_{j,l}$.

By including the extraordinary orthogonal groups $O^- (n, F_q)$ (= automorphisms of the quadratic space $(F_q^n, Q^-)$ where $Q^- (x_1, \ldots, x_n) = \mu x_1^2 + x_2^2 + \cdots + x_n^2$ and $\mu \in F_q$ is a nonsquare) the relations of Theorem 3 simplify considerably. Since $O^- (1, F_q) = \mathbb{Z}_2$, let $v_k^- \in H_k(BO(1, F_q), \mathbb{Z}_2) = \mathbb{Z}_2$ denote the generator. Let $O^+ (n, F_q) = O(n, F_q)$ (= automorphisms of the quadratic space $(F_q^n, Q^+)$ where $Q^+ (x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$) and set $v_k^+ = v_k$. There are isomorphisms $(F_q^n, Q^-) \oplus (F_q^n, Q^+) \approx (F_q^{n+m}, Q^+)$.

**Theorem 4.** As an algebra

$$H_*(BO(0, F_q), \mathbb{Z}_2) \oplus \bigoplus_{n=0}^{\infty} H_*(BO(n, F_q), \mathbb{Z}_2)$$

is commutative with generators $v_i^+, v_i^-$, $i \geq 0$ subject to the single relation $(v_i^+)^2 = (v_i^-)^2$.

The generator $v_{i,j}$ of Theorem 3 is seen to decompose as $v_{i,j} = v_i^- v_j^-$. Proofs and additional details will appear in our forthcoming paper.

**References**


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