DEFORMING P.L. HOMEOMORPHISMS
ON A CONVEX 2-DISK

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1. The main result. Let $D$ be a convex disk in $\mathbb{R}^2$ whose boundary is a polygon. By a triangulation of $D$, we mean a (rectilinear) simplicial complex which has $D$ as its underlying space. We shall call a homeomorphism $f$ of $D$ onto $D$ a p.l. homeomorphism if there exists a triangulation $K$ of $D$ such that the restriction of $f$ to each simplex $\sigma$ of $K$ is a linear map of $\sigma$ into $\mathbb{R}^2$. We shall consider only those p.l. homeomorphisms of $D$ which are pointwise fixed on the boundary of $D$. In this note, we announce the following result.

**Theorem A.** For each p.l. homeomorphism $f$ of $D$, there exists a triangulation $K$ of $D$ such that $f$ may be realized by successively moving the vertices of $K$ in a finite number of steps (with the motion being extended linearly to each simplex of $K$) such that in the process of moving, none of the simplices is allowed to collapse.

The general problem of deforming a prescribed map of a space into the identity map, or vice versa, in a specific manner has a long history. For the special case of deforming a particular homeomorphism of an $n$-cell into the identity map through a special class of homeomorphisms, H. Tietze proved as early as 1914 that any homeomorphism of a 2-disk, which is pointwise fixed on the boundary of the disk, can be deformed into the identity map through a family of such homeomorphisms [5]. This result was extended in 1923 for an $n$-dimensional cell by J. W. Alexander [1]. The technique used by Alexander can in fact be used to show that each p.l. homeomorphism on a polyhedral $n$-cell, which is pointwise fixed on the boundary of the cell, can be deformed into the identity map through a family of such p.l. homeomorphisms. However, each of the p.l. homeomorphisms of the family requires a different triangulation of the domain space. It is therefore natural to ask whether this

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deformation can be carried out in such a way that all the p.l. homeomorphisms required in the deformation process are linear with respect to a fixed triangulation of the n-cell. Our Theorem A clearly answers this question in the affirmative for a convex 2-dimensional polyhedral disk. Our proof of Theorem A, as outlined in the next two sections, is based on the assumptions that the disk is convex and 2-dimensional. We do not know whether the theorem is still true for a higher dimensional disk or for a 2-dimensional disk which is not convex.

2. Preliminaries. For each triangulation $K$ of $D$, we shall let $L(K)$ be the space of all p.l. homeomorphisms of $D$ which are linear with respect to $K$. The space $L(K)$ is equipped with the compact open topology. Observe that each element $f \in L(K)$ is completely determined by the image under $f$ of the vertices of $K$ which are contained in the interior of $D$. Thus, if an ordering, say from 1 to $n$, is assigned to these interior vertices of $K$, each element $f \in L(K)$ may be identified as a point in the space $\mathbb{R}^{2n}$. In fact, one may establish without too much effort

**Proposition 1.** For each triangulation $K$ of $D$, the space $L(K)$ may be identified as an open subset of $\mathbb{R}^{2n}$ where $n$ is the number of vertices of $K$ contained in the interior of $D$.

Under this identification of $L(K)$ as an open subset of $\mathbb{R}^{2n}$, we observe that each element $f \in L(K)$ has a neighbourhood $N$ in $L(K)$ such that each $g \in N$ can be obtained from $f$ by successively moving the images $f(v)$ of the vertices $v$ of $K$. To see this, one needs only to construct a "cubic box" centered at $f$ in $\mathbb{R}^{2n}$ which is contained in $L(K)$. Then one may deform $f$ to any other element $g$ of the box by moving successively the component of $f$ in each copy of $\mathbb{R}^2$ to the corresponding component of $g$. From this observation and an elementary compactness argument, one establishes immediately

**Proposition 2.** Let $K$ be a triangulation of $D$ and let $f, g$ be two elements of $L(K)$. The element $f$ may be deformed to $g$ by moving successively the images $f(v)$ of the vertices of $K$ if and only if $f$ may be connected to $g$ by a path in the space $L(K)$.

With this proposition, one may rewrite our Theorem A in the following equivalent form.

**Theorem B.** For each p.l. homeomorphism $f$ of $D$, there exists a triangulation $K$ of $D$ such that $f \in L(K)$ and $f$ may be connected to the identity map of $D$ by a path in $L(K)$. 
3. Sketch of the proof of Theorem B. Let $K$ be a triangulation of $D$. A vertex of $K$ is called a boundary vertex if it is contained in $\text{Bd}(D)$. The triangulation $K$ will be called a proper triangulation if no three boundary vertices of $K$ are on a straight line. Intuitively, a proper triangulation of $D$ has no vertices on the sides of $D$ except at the “corners” of $D$. We first establish a special case of Theorem B, the case when the p.l. homeomorphism $f$ belongs to $L(K)$ for a proper triangulation $K$ of $D$ (cf. [2]).

**Proposition 3.** The space $L(K)$ is pathwise connected for a proper triangulation $K$ of $D$.

We may then use an argument similar to that described in [3] to show that for each p.l. homeomorphism $f$ of $D$, there is a proper triangulation $K$ of $D$ such that $f$ may be connected to $L(K)$ by a path in some larger space $L(K')$. This implies Theorem B. The details of all the proofs will appear in [4].

**References**


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