EXTENSION THEOREMS FOR REDUCTIVE GROUP ACTIONS ON COMPACT KAHLER MANIFOLDS

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Let \( G \) be a connected complex reductive Lie group. Noting [3], [7], [8] that \( G \) has the structure of a linear algebraic group, let \( \widetilde{G} \) be any projective manifold in which \( G \) is Zariski open and which induces the above algebraic structure on \( G \). The purpose of the present note is to announce

**Proposition I.** Let \( G \) be as above and act holomorphically on a compact Kaehler manifold \( X \). Assume that the Lie algebra of holomorphic vector fields on \( X \) generated by \( G \) is annihilated by every holomorphic one form.

Let \( \Phi: Y \to X \) be a holomorphic map where \( Y \) is a normal reduced analytic space. Consider the equivariant map \( \Phi': G \times Y \to X \); \( \Phi' \) extends meromorphically (in the sense of Remmert) to \( \widetilde{G} \times Y \).

**Remarks.** The condition on vector fields annihilated by one forms is automatically satisfied if (cf. [12]–[14]) \( H^1(X, \mathbb{Q}) = 0 \), or \( G \) is semisimple, or if every generator of the solvable radical of \( G \) has a fixed point, or if \( G \) is a linear algebraic group acting algebraically on a projective \( X \). Taking \( Y \) to be a point, one gets the orbits of \( G \) to be Zariski open in their closures which are analytic sets. A simple corollary is the classical result that there is only one structure of a linear algebraic group on \( G \) (cf. [7]), and in fact any reductive connected subgroup of an algebraic group over \( \mathbb{C} \) is an algebraic subgroup.

As a further application of the techniques used, a new proof of an improved form of a fixed point theorem (cf. [12], [13], [14]) of the author is given:

**Proposition II.** Let \( S \) be a connected solvable Lie group acting holomorphically on a compact Kaehler manifold \( X \). The following are equivalent:

(a) \( S \) has a fixed point on \( X \).


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(b) $S$ leaves a compact set in a fibre of the Albanese map invariant.

c) $S$ has a fixed point within any compact set $K$ on $X$ that $S$ leaves invariant.

d) The Lie algebra of vector-fields that $S$ generates on $X$ is annihilated by every holomorphic one form on $X$.

REMARK. The assertion (c) where $K$ is any compact set is new; the method of proof allows one to relax the compactness of $X$ and show if, in addition, $H^1(X, \mathcal{O}_X) = 0$, then (c) is true.

The following is the fundamental observation on which everything rests.

**Lemma.** Let $X$ be a compact Kaehler manifold and $\rho: \mathbb{C}^* \to \text{Aut}(X)$ a holomorphic $\mathbb{C}^*$ action that has at least one fixed point. Let $A: \mathbb{C}^* \to X$ be a holomorphic equivariant map onto an orbit: then $A$ extends to a homomorphic equivariant map $\tilde{A}$ of $\mathbb{C}P^1$ to $X$.

**Proof.** Assume without loss of generality that $A(\mathbb{C}^*)$ is not a point. Let $\mu$ be a Kaehler metric on $X$ and $\omega$ the associated Kaehler form. Assume that $\mu$ has been averaged with respect to the circle subgroup $S^1 \subseteq \mathbb{C}^*$. Let $\chi$ be the holomorphic vector-field on $X$ associated to $\rho: \mathbb{C}^* \to \text{Aut}(X)$.

Because of equivariance, the Jacobian, $dA$, of $A$, maps some constant multiple of $z(\partial/\partial z)$ onto the restriction of the vector-field $\chi$ to $A(\mathbb{C}^*)$. Without loss of generality this constant is assumed to be one.

Let $A^*\mu = a(r) \, dz \otimes d\bar{z}$ where $a(r)$ is positive and depends only on $r$ due to the $S^1$ averaging of $\mu$. $A^*\omega = (i/2)a(r) \, dz \wedge d\bar{z}$.

$$
\mu(\chi, \chi) = \mu \left( dA \left( z \frac{\partial}{\partial z} \right), dA \left( z \frac{\partial}{\partial z} \right) \right) = A^*\mu \left( z \frac{\partial}{\partial z}, z \frac{\partial}{\partial z} \right)
$$

$$
= a(r)|z|^2 \leq M < \infty
$$

where $\sup_x \mu(\chi, \chi) = M < \infty$.

Now by Lichnerowicz [5] there exists a $C^\infty$ function $\phi$ on $X$ such that $\bar{\partial}\phi = \omega(\chi)$. Pulling back and, without confusion, letting $\phi$ stand for $A^*\phi = \phi(A(z))$, one has

$$
\frac{i}{2} z a(r) \, d\bar{z} = \frac{\partial \phi}{\partial z} \, d\bar{z} \quad \text{or} \quad \frac{i}{2} z a(r) = \frac{\partial \phi}{\partial z}.
$$

Now fix one circle, say the unit circle $C_1 \subset \mathbb{C}^*$ and let $C_R = \{ z \in \mathbb{C}^* | \, |z| = R \}$. Assume $R > 1$; $C_1$ and $C_R$ bound an annulus $\tilde{A}$ with $\partial \tilde{A} = C_R - C_1$. Now
\[ \int_{A} A^* \mu = \int_{A} \int_{z} \frac{i}{z} a(r) dz \wedge d\bar{z} = -\int_{A} \int_{z} \frac{\partial \phi}{\partial z} \frac{d\bar{z} \wedge dz}{z} \]

\[ = -\int_{C_R} \phi \frac{dz}{z} + \int_{C_1} \phi \frac{dz}{z} = \frac{1}{i} \int_{0}^{2\pi} \phi(Re^{i\theta}) \, d\theta - C \]

with C a constant. Now \(|\int_{0}^{2\pi} \phi(Re^{i\theta}) \, d\theta| \leq M' < \infty\) since \(\phi\) is the pullback of a bounded function on \(X\).

Therefore \(\int_{A} A^* \mu \leq M'' < \infty\) where \(M''\) is a positive constant independent of \(R\). Thus by Bishop's extension theorem (cf. [1], [2]), \(A\) extends holomorphically over \(\infty\). An identical argument gives extension at 0. Q.E.D.

Using the above Lemma and the Levi-Griffiths-Shiffman-Siu extension theorem (cf. [2], [9], [10], [11]) repeatedly, one proves the result for \(SL(2, \mathbb{C})\) and groups of the form \((\mathbb{C}^*)^n\) that have a fixed point on \(X\). Then one proves it for one parameter unipotent subgroup of \(G\) by using the above \(SL(2, \mathbb{C})\) result on an \(SL(2, \mathbb{C})\) in \(G\) containing the subgroup; this can be done by Jacobson-Morosow (cf. [4]). One now proves it for a Borel subgroup of \(G\) and uses an argument depending on the fact that one has a locally trivial fibering of \(G\) over \(G/B\) which is compact.

In the very interesting paper [6] of Lieberman, related matters are discussed.

**BIBLIOGRAPHY**


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