ON CUSPIDAL REPRESENTATIONS OF
p-ADIC REDUCTIVE GROUPS

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Abstract. Let \( k \) be a \( p \)-adic field, and \( G \) a reductive connected algebraic group over \( k \). Fix a maximal torus \( T \) of \( G \) which splits in an unramified extension of \( k \), and which has the same split rank as the center of \( G \). For each character \( \theta \) of \( T(k) \), satisfying some conditions, there is a cuspidal representation \( \gamma_\theta \) of \( G(k) \) which is a sum of a finite number of irreducible representations; the correspondence \( \theta \mapsto \gamma_\theta \) is one-to-one on the orbits of such characters by the little Weyl group of \( T \); furthermore, the formulas for the formal degree of \( \gamma_\theta \) and its character for sufficiently regular elements of \( T(k) \) are given: they are formally the same as is the discrete series for real reductive groups.

1. Unramified maximal tori. Let \( k \) be a \( p \)-adic field, that is a finite extension of \( \mathbb{Q}_p \) or a field of formal series over a finite extension of \( \mathbb{F}_p \). We denote by \( \overline{k} \) the residue field of order \( q \).

Let \( G \) be a reductive connected algebraic group defined over \( k \), the derived group \( G_{\text{der}} \) of which is simply connected. A maximal torus of \( G \) defined over \( k \) is called minisotropic if it normalizes no (proper) horocyclic subgroup of \( G \) defined over \( k \).

Lemma. Suppose there exists a minisotropic maximal torus \( T \) of \( G \) which splits in a finite unramified extension \( L \) of \( G \). Then the Galois group \( \Gamma \) of \( L \) over \( k \) has a unique fixed point \( v \) in the apartment of \( T \) in the building of \( G_{\text{der}}(L) \) \([2]\); moreover, the face of \( v \) is minimal amongst the faces in this apartment which are invariant by \( \Gamma \).

2. Characters. We conserve notations and hypotheses of §1 and the Lemma. Let \( \theta \) be a continuous character of \( T(k) \). For each \( \lambda \in X^0(T) \), the lattice of rational one-parameter subgroups of \( T \), we define a character \( \theta_\lambda \) of \( L^x \) by

\[ \theta_\lambda(x) = \theta_\lambda(x) \]


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\[ \theta_\lambda(z) = \theta\left( \prod_{\Gamma} \left( z^{\lambda} \right) \right), \quad z \in L^x. \]

**Definition 1.** The character \( \theta \) is called regular if, for every root \( \alpha \) of \( (G, T) \), the character \( \theta_{\alpha^v} \) of \( L^x \) associated to the coroot \( \alpha^v \) is nontrivial.

For each root \( \alpha \), we denote by \( |\alpha|_\theta \) the conductor of \( \theta_{\alpha^v} \) and let \( R_f \) be the set of roots of \( (G, T) \) such that \( |\alpha|_{\theta} \leq f \).

**Definition 2.** The character \( \theta \) is called good if it is regular and if, for every \( f \), the set \( R_f \) is a convex set of roots.

We need the following form of Macdonald's conjecture (cf. [1, C-6.7]):

Let \( \bar{S} \) be a reductive connected algebraic group over \( \bar{k} \), \( \bar{T} \) a minisotropic maximal torus of \( \bar{S} \); fix a finite extension \( \bar{L} \) of \( \bar{k} \) which splits \( \bar{T} \); let \( \Gamma \) be its Galois group. A character \( \bar{\theta} \) of \( \bar{T}(\bar{k}) \) is called regular if, for every root \( \alpha \) of \( (\bar{S}, \bar{T}) \), the character \( z \mapsto \bar{\theta}(\prod_{\Gamma} (z^{\alpha^v})) \) of \( \bar{L}^x \) is nontrivial; if \( \bar{\theta} \) is a regular character of \( \bar{T}(\bar{k}) \), then there exists a unique class \( \bar{\sigma}_g \) of representations of \( \bar{S}(\bar{k}) \) such that, if \( S \) denotes the Steinberg representation of \( S(\bar{k}) \), \( S \otimes \bar{\sigma}_g \) is the induced representation \( \text{Ind}(S(\bar{k})), \bar{T}(\bar{k}), \bar{\theta}) \). Moreover \( \bar{\sigma}_g \) is cuspidal, and the intertwining number of two such representations \( \bar{\sigma}_{g_1} \) and \( \bar{\sigma}_{g_2} \) is equal to the number of elements in the little Weyl group of \( T \) in \( G(\bar{k}) \) which send \( \bar{\theta} \) on \( \bar{\theta} \).

3. Cuspidal representations.

**Theorem.** Let \( G \) be a reductive connected algebraic group over the \( p \)-adic field \( k \), the derived group \( G_{\text{der}} \) of which is simply connected. Let \( T \) be a minisotropic maximal torus which splits in an unramified extension of \( k \). Fix a good character \( \theta \) of \( T(k) \). Assume Macdonald's conjecture and one of the following conditions:

(i) \( |\alpha|_\theta > 1 \) for every root \( \alpha \) of \( (G(L), T(L)) \)

(ii) the residual characteristic of \( k \) is not 2 and there exists a rational representation \( \rho \) of \( G_{\text{der}} \) such that the corresponding bilinear form \( \text{Tr} \rho(X)\rho(Y) \) on the Lie algebra \( \text{Lie } T_{\text{der}}(\bar{k}) \) is nondegenerate.

Then there exists a class \( \gamma_\theta \) of representations of \( G(k) \) such that:

(a) \( \gamma_\theta \) is a finite sum of irreducible representations of \( G(k) \), the coefficients of which have compact support modulo the center;

(b) the intertwining number of two such representations \( \gamma_{\theta_1} \) and \( \gamma_{\theta_2} \) is the number of elements in the little Weyl group \( W(T) \) of \( T \) in \( G(k) \) which send \( \theta_1 \) on \( \theta_2 \);
(c) there exists a Haar measure on $G(k)$, independent of $\theta$, such that the formal degree of $\gamma_\theta$ is

$$d(\theta) = \left( \prod_R q^{\vert \alpha \vert_\theta - 1} \right)^{1/6}$$

where $R$ is the set of roots of $(G, T)$;

(d) for $t \in T(k)$ such that $\mathrm{val}(t^\alpha - 1) \geq |\alpha|_\theta / 3$ for every $\alpha \in R$, the value on $t$ of the character of $\gamma_\theta$ is given by

$$\mathrm{Tr} \, \gamma_\theta(t) = (-1)^{l(G)}(-1)^{\sum_R \Gamma^{(\alpha_\theta - 1)}} \sum_{w(T)} \frac{\theta}{\Delta} (w t)$$

where $l(G)$ is the split semisimple rank of $G$, $\Delta$ is the $W(T)$-invariant function on the regular elements of $T(k)$ given by

$$\Delta(t) = (-1)^{\sum_R \Gamma^{\mathrm{val}(t^\alpha - 1)}} |\mathrm{Det}_{\mathrm{Lie}} G/\mathrm{Lie} T(\mathrm{Ad} \, t - 1)|^{1/5},$$

and $\Gamma$ is the Galois group over $\overline{k}$ of an unramified extension which splits $T$.

4. Remarks. 1. The proof is based upon an explicit construction of $\gamma_\theta$ (assuming Macdonald's conjecture), obtained by inducing a finite dimensional representation of a compact open subgroup of $G(k)$ naturally associated to $T$ and $\theta$; the essential tool is given by Weil's paper about Heisenberg groups [5]; we used too an argument given by R. Howe [4].

2. In the case where $|\alpha|_\theta$ is constant and strictly greater than 1, and if the point $v$ of the lemma is special, the proofs are given in [3].

3. G. Lusztig has just proved Macdonald's conjecture.

REFERENCES


4. R. Howe, Tamely ramified supercuspidal representations of $GL_n$ (preprint).


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