bundle are defined using the Chern classes of the complexification of the vector bundle. These are integral characteristic classes in dimensions divisible by 4. With this, all of the characteristic classes to be considered in the book have been defined.

The rest of the body of the text is devoted to applications of the characteristic classes. First is an application to partial calculation of the cobordism groups. It is shown that the vanishing of all Pontrjagin numbers of a smooth compact oriented manifold are necessary conditions for the manifold to be a boundary. This leads to the study of the oriented cobordism ring and to a calculation of the tensor product of this ring with the rationals.

Next is a proof of the signature theorem using multiplicative sequences. This theorem expresses the signature of a smooth compact oriented manifold in terms of Pontrjagin numbers of M. It is applied to prove that the rational Pontrjagin classes of a smooth compact oriented manifold are piecewise linear invariants of the manifold. In fact, rational Pontrjagin classes are defined for compact rational homology manifolds and these combinatorial Pontrjagin classes are shown to agree with the differentiable ones for smooth manifolds. As one application of this, it is shown that for every dimension $\geqq 8$ there are two smooth simply-connected manifolds having the same homotopy type but not piecewise linearly homeomorphic. Another application is to provide an example of a triangulated 8-dimensional compact manifold having no compatible smooth structure.

In addition to those results dealing directly with characteristic classes the book contains a discussion of various topics of interest in algebraic and differential topology which are relevant to the text. The epilogue is a brief survey of results since the original notes appeared. It has three parts, one devoted to generalizations of the theory to nonsmooth manifolds, one devoted to the theory for smooth manifolds with additional structure, and one devoted to generalized cohomology theories.

As can be seen from this discussion of the contents, there is a lot of mathematics included in the book. It is a valuable and welcome addition to the literature.

E. H. SPANIER

Lie groups, Lie algebras, and their representations, by V. S. Varadarajan, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974, xiii+430 pp., \$19.95

Compact Lie groups and their representations, by D. P. Želobenko, Translations of Mathematical Monographs, Vol. 40, American Mathematical Society, Providence, R.I., 1973, viii+448 pp., \$35.70

The books by Varadarajan and Želobenko are surprisingly dissimilar. Both are written as detailed mathematical expositions of the theory of representations of compact Lie groups. But Varadarajan's is written with mathematics students in mind, whereas Želobenko's is influenced by the needs of theoretical physicists.

Varadarajan's book began as a set of lecture notes in a course on Lie groups and is intended to "introduce the aspiring graduate student as well as the nonspecialist mathematician to the fundamental themes of the subject." Its spirit is captured by the Sanscrit quotation at the beginning of the preface, which the author has kindly translated for the reviewer as an invocation to the sun in search of enlightenment.

Želobenko's book began as a series of lectures given to a group of theoretical physicists. But the author resisted the temptation to communicate results without methods and instead "endeavored throughout to underline the internal coherency of the theory."

A summary of the contents of Varadarajan's book is in order. There are four chapters. Chapter 1 is about manifolds and corresponds roughly to the third chapter of Chevalley [**2**]. But it contains also enough discussion of differential forms to talk about integration, and it includes the most preliminary facts about linear differential operators. This all takes 40 pages. Chapter 2 is entitled "Lie groups and Lie algebras" and corresponds to an expanded version of Chapter 4 of Chevalley's book or to Chapter 2 of Helgason [**4**]. It is expanded by the presence of examples, by Taylor series expansions, and by an elegant analytic proof of the Campbell–Hausdorff formula, adapted from lectures by Bargmann. The first two chapters can form a one-semester course. Chapter 3 is mostly about structure of Lie algebras and includes discussion of the universal enveloping algebra, along with the basic theorems of Lie, Engel, and Ado and with Cartan's criteria and the Levi decomposition. Chapter 4 is the part of the book about representations. It starts with Cartan subalgebras, passes through the unitary trick, and concludes with the Weyl character formula and the dimension formula.

The book by Varadarajan is an outstanding one in many ways. The author never deviates from his single purpose of writing for graduate students and nonspecialists. There are no lapses into encyclopedic detail, and the book proceeds in logical fashion, its path determined by the principal theorems.

This reviewer, fresh from teaching a course on this material (unfortunately without the benefit of this book), is aware of several tricky points in expositions of representations of compact Lie groups, and Varadarajan has handled all but one of them admirably. A discussion of them will illustrate how much effort has gone into the writing. One should bear in mind that the problems that all authors have to face in deciding where to begin and what to assume are especially acute in Lie groups. The difficulty is that many of the techniques and lesser results of a first-semester course in Lie groups are irrelevant for later study. In fact, they are even misleading, since they lack the neatness and elegance of most of the subject. Also, the later material requires a considerable amount of algebra and analysis as prerequisites, by graduate student standards, and it is not feasible to develop all this material if one ever wants to get to the subject at hand. An author must handle these problems with some delicacy in the course of writing a worthwhile book about Lie groups. A proof done with theorems from outside Lie groups may be a little shorter, but a proof done by group-theoretic methods may both illustrate useful techniques and avoid a long development of preliminary material.

The first tricky point is in the proof of the existence of an analytic subgroup corresponding to each Lie subalgebra. Most other books leave a major gap in the proof, failing to deduce correctly the continuity of multiplication in the subgroup from the continuity of multiplication as a mapping into the whole group. The books by Helgason [**4**] and Chevalley [**2**] are exceptional in giving correct proofs;[1] Helgason's is by using a weakened form of the Campbell–Hausdorff formula (see [**4**, p. 99, Formula (14)]), and Chevalley's is by a direct attack through a separability result for submanifolds (see [**2**, p. 96, Proposition 2]). Varadarajan follows Chevalley, except that he restricts attention to integral submanifolds and is therefore able to eliminate a lemma of Chevalley's that becomes superfluous, and there are no mistakes.

A second tricky point occurs in the proof of Cartan's criterion for solvability. There is a nice proof if the underlying field is a subfield of the complex numbers, but Varadarajan does not give this proof since he wants to allow arbitrary fields of characteristic 0. For such general fields one can either quote some results about real closed fields and copy the nice complex-number proof with no essential change,[2] or else one can use a complicated argument that rests on linear functionals on the field regarded as a vector space over the rationals. Varadarajan does the latter. The reviewer prefers to use the simpler argument for the simpler fields and to point out to algebraists how to use real closed fields to handle the general case.

A third point is the theorem that the exponential map is onto for a compact connected Lie group. Usually this theorem is proved as a consequence of some facts in Riemannian geometry and forces an author either to make a considerable digression or to assume too much of the graduate student reader. Adams [**1**] avoided Riemannian geometry in his book for topologists and gave Weil's proof based on the Lefschetz fixed-point formula. Varadarajan has found what seems to this reviewer the best proof yet; it is direct and group-theoretic and proceeds by induction on the dimension of the group, very much in the spirit of many arguments given by Harish-Chandra.

A fourth tricky point is the proof of Weyl's theorem that the universal covering group of a compact connected semisimple group is compact. A standard proof uses Riemannian geometry. But this time if one is willing to use the Peter-Weyl theorem, the proof quickly reduces by group theory to showing the fundamental group is finitely generated.[3] One can conceive of a number of complicated ways of proving the finite generation, but Varadarajan has a simple one that takes only a dozen lines.

A fifth and final point is the result that an algebraically integral dominant linear form is a highest weight if the compact Lie group is simply-connected. One can skip over this theorem, but the theory suffers if one does. Or one can use the classification of complex semisimple Lie algebras, following Cartan,

[1] The reviewer does not claim that no other book gives a correct proof.

[2] This possibility was pointed out to the reviewer by R. Scott Fowler.

[3] Varadarajan does not use the Peter-Weyl theorem in his reduction argument, but the proof is only a little more complicated without this theorem.

doing painful computations in the cases of E_6 and E_7. Or one can handle the problem right and give a proof without the classification. Varadarajan handles it right and gives the proof due to Chevalley and Harish-Chandra that an algebraically integral dominant form is the highest weight of a representation of a complex semisimple Lie algebra.

These details give some indication how extraordinarily polished and carefully written the book is. The reviewer found a couple of misprints in internal references to theorems but no mistakes of substance. The book contains many examples. A particularly nice one, mentioned earlier, is the example, following the definition of Lie group, of an algebraically defined group of matrices. It is easier to prove and understand that such a group is a Lie group than it is to prove and understand that a closed subgroup of a Lie group is a Lie group. In addition, the book contains many exercises, which the student should not skip.

As for prerequisites, the book makes what seems a reasonable compromise. Material that people ought to learn before graduate school is taken for granted. This includes topological groups and the Baire category theorem. The necessary theorems on analytic differential equations are proved in an appendix to Chapter 1. Results on covering spaces and the fundamental group are summarized on pp. 61–63. The required theory of a single linear transformation is given on pp. 149–166, partly with proofs and partly with no proofs but with references. The last sections of Chapter 4 assume the Frobenius reciprocity theorem (reference given) and the Peter-Weyl theorem (no reference given).

There is little to find fault with. One criticism is that it is a little hard to distinguish at a glance the really important theorems from the others. Another is that the reader with some knowledge of the subject may be misled by one or two terms if he begins in the middle and is not wary: "regular imbedding" is a term to watch out for, and a "nilpotent Lie algebra" for Varadarajan turns out to be a standard nilpotent Lie algebra because of Engel's theorem; a "manifold" is not a manifold until it has a countable base. One more cirticism is that the author's success at making the proof of the Frobenius theorem on involutive distributions as short as possible sacrifices the chance to give the reader a real understanding of the exponential mapping in terms of differential equations. Finally this reviewer objects on principle to long dry sections of preparatory material such as the one here on linear algebra on pp. 149–166. (Partly this section suffers because of preparation for the proof mentioned above as the "second tricky point.") The author's theory perhaps is that the rug is neater if the dirt underneath it is all in one pile.

Turning to Želobenko's book, one finds something quite different. The book was written for theoretical physicists, and it will emerge below that the book is useful also for specialists in Lie groups. In the preface one reads, "The author is fully aware that the book is to a significant degree a set of lecture notes; accordingly it lays no claims to be a textbook or systematic monograph." Indeed, a mathematics graduate student would be ill advised to learn the field from this book.

Želobenko's book is divided into three parts, each consisting of a number of chapters. Part I consists of preliminaries and motivation—a quick trip through a first semester course in Lie groups, a smattering of Riemannian geometry and tensors, and some soft material on group representations. Part II, as will be seen below, is the core of the book: After a chapter on the Peter-Weyl theorem, Želobenko deals with the unitary group $U(n)$ and the special unitary group $SU(n)$. There are weights and roots, models for the irreducible representations, Young diagrams, Casimir operators, both forms of the Weyl character formula (as a quotient and as an integer combination of exponentials), and results on reductions of tensor products. Part III, entitled "General theory," deals with general compact connected Lie groups. To a degree it runs parallel to Part II, but not everything generalizes well. The book concludes with three appendices, the third of which is on elementary-particle theory and is the most notable. Unfortunately this reviewer found the third appendix a bit murky.

To make some judgment about the suitability of Želobenko's book for physicists, it is necessary to have at least a hazy understanding of the physics involved. This discussion will be limited to that part of physics in which the representations of $SU(n)$ play the most significant role, namely the study of nuclear interactions in elementary-particle theory.

In quantum mechanics the things that one can observe from experiments are eigenvalues (or the spectrum, if there are not discrete eigenvalues) of certain selfadjoint operators on suitable Hilbert spaces. Physicists are particularly conscious of conservation laws, which correspond to selfadjoint operators whose associated one-parameter groups of unitary transformations commute with the Hamiltonian. Thus a system of conservation laws leads to a group of symmetries, namely a group of unitary operators commuting with the Hamiltonian.

Elementary-particle theory is plagued by something called approximate conservation laws. That is, it is observed experimentally that it is possible to attach numbers to various particles in such a way that the sum over all particles is conserved during an interaction in an accelerator, provided the interactions are all of the right kind. The kinds of forces are classified, in increasing intensity, as gravitational, weak, electromagnetic, and strong. Weak interactions typically involve neutrinos, electromagnetic interactions typically involve photons, and strong interactions involve nuclear forces. For the discussion at hand, the numbers attached to particles are called isospin and hypercharge, and the interactions of the right kind are the strong interactions.[4] In practice, one cannot obtain a pure separation of the kinds of forces (no one has seen an uncharged proton) and has to make do. One theoretical problem is to account for the approximate conservation of isospin and hypercharge by a group of symmetries.

Some experimental facts that one knows are that the different nuclear particles seem to fall into sets of 8 and sets of 10, with the particles within a

[4] Another number is called baryon number, which is conserved in all interactions.

set having approximately the same mass and having the same baryon number. A further fact is that at certain energies the scattering angles seen in scattering an electron and a proton suggest that a proton is not a single particle but may be built from three (or maybe more[5]) constituents.

The first facts are accounted for nicely by identifying the particles in a set with a basis of weight vectors for an irreducible representation of $SU(3)$. The weights, which can be regarded as the pair of eigenvalues for the action of a basis of the diagonal subalgebra on the representation space, are just the isospin and hypercharge. The $SU(3)$ representation does not classify the particles, since it deals only with two conserved quantities; the proton-neutron "octet" goes with the same 8-dimensional representation as the antiproton-antineutron "octet." Nor does every representation occur in the theory. Higher-dimensional irreducible representations should be related to more complicated nuclei than single particles, but the evidence in support of this hope is still scanty. On the other hand, although a particle has been observed that goes with the trivial representation, none has been observed that goes with either of the two 3-dimensional representations (which are called $\mathbf{3}$ and $\bar{\mathbf{3}}$).

According to the theory, the phenomena that can occur in strong interactions are limited by what happens in the reduction of the tensor product of the representations corresponding to the incoming particles or nuclei. Therefore knowledge of the irreducible constituents of tensor products and how the weights combine is fundamental for theoretical physics.

This interpretation of tensor products is used, somewhat tentatively, as a basis for explaining away the second experimental fact, that the proton does not appear to be a point mass. It is postulated that there are particles going with the two representations $\mathbf{3}$ and $\bar{\mathbf{3}}$, 3 kinds of quarks and 3 kinds of antiquarks. A proton is 3 quarks, in the sense that a proton occurs in the reduction of $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$. And so on. This part of the theory still has unsatisfactory aspects.

Returning to Želobenko's book, one sees why Part II is so central. Physicists use primarily $SU(2)$ and $SU(3)$. (For a while there was a theory built around $SU(6)$.) In addition to knowing about weights and the parametrization of irreducible representations, they need facility at reducing tensor products. Identifying the irreducible constituents and their weights in a tensor product is most important; finding bases for the constituents is still important but secondary.

Part III, however, is not irrelevant. The search goes on in physics for the right group to describe everything at once, and familiarity with a range of groups other than $SU(n)$ is useful.

As it happens, the techniques and facts required by physicists about $SU(n)$ are some of the same ones required by mathematicians who specialize in Lie groups and do elaborate private computations that never appear in their published papers. For this reason, Želobenko's book will be of interest to specialists in Lie groups.

[5] Some experimental evidence from Fall 1974 suggests four constituents.

Mastering techniques for manipulating representations of $SU(n)$ requires learning both how and why. Želobenko's book satisfactorily answers only "why" and therefore cannot stand alone. The primary tool is Young diagrams, from which one can read off weights, dimension formulas, and reductions of tensor products. None of the techniques for extracting this information from Young diagrams is made explicit in the book, although the theorems do prove that the techniques work (once one knows what they are). When Želobenko's book is supplemented by a book, such as the one by Lichtenberg [5], that makes the techniques explicit, the pair of books becomes a useful source of information that is otherwise hard to come by.

The book contains a substantial amount of material on realization of representations that is reproduced from the author's own research papers. The bias toward this material, which is very satisfactory from a theoretical standpoint, provides a bias away from explicitly computable results and may account for the omissions with Young diagrams. However, the bias does increase the theoretical interest of the book to mathematicians who specialize in Lie groups.

True to the disclaimer in the preface, Želobenko has not written a polished book. A first defect is that the book does not quite know where to begin. The first section gives a careful definition of group, but later sections indicate that the reader is really supposed to know about manifolds and Lie groups already. The assumption that the reader has already seen some Riemannian geometry and just needs a few reminders may not be unreasonable for theoretical physicists.

By way of comparison with Varadarajan's book, here is how Želobenko handles the tricky points: The existence of analytic subgroups corresponding to Lie subalgebras is stated but not proved. Cartan's criterion for solvability is proved nicely by an argument due to Dynkin that uses the already-proved existence of Cartan subalgebras in arbitrary Lie algebras. The proof that the exponential is onto is the standard one by Riemannian geometry. Weyl's theorem is given a geometric proof that omits any justification for the continuous selection of diagonalizing elements. The existence of representations for a simply-connected compact group corresponding to all algebraically integral dominant linear forms is proved by working with the complexified group and building the representation globally by means of the Bruhat decomposition[6] and the author's own theorems on indicator systems; this approach is new to the reviewer, but, in retrospect, it is a natural one. In short, the author's success in handling these points has been uneven.

The frequent misprints in the book are beyond the author's control, since the book is a translation. A more serious matter is the occasional substantive mistakes and misleading proofs. In the category of mistakes are the following assertions: (1) (see p. 24) In any one-sided invariant Riemannian metric on a Lie group, the one-parameter subgroups are geodesics. (2) (see p. 291) Any two points in a Riemannian space can be connected by at least one geodesic.

[6] The full decomposition is often called the Bruhat decomposition, but the author uses only the open cell and calls the result the Gauss decomposition.