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AN ITERATIVE SOLUTION OF A VARIATIONAL INEQUALITY FOR CERTAIN MONOTONE OPERATORS IN HILBERT SPACE

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Let $A$ be a multivalued monotone operator on a real Hilbert space $H$ and let $C$ be a nonempty closed convex subset of $D(A)$. If $f \in H$, by a solution of the variational inequality

\[(z_0 - f, x - u_0) \geq 0 \quad \forall x \in C,\]

we mean a pair (or, sometimes, just the first component of a pair) $[u_0, z_0] \in A$ satisfying (1) such that $u_0 \in C$. We denote the set of solutions $u_0$ by $E$. We shall assume the existence of a solution of (1) and show how to construct it as the weak limit of a sequence $\{x_n\}$ satisfying

\[x_{n+1} = P(x_n - t_n(v_n - f)), \quad v_n \in Ax_n,\]

where $\{t_n\} \subset [0, \infty)$ and $P$ is the proximity mapping of $H$ onto $C$. For conditions sufficient to guarantee $E \neq \emptyset$, see Browder [4], Lions [10].

**Theorem 1.** Suppose there exists $u_0 \in E$ such that

\[\{(v - f, x - u_0) = 0, x \in C, v \in Ax\} = x \in E.\]

If, in (2), $\Sigma t_n = \infty$, $\Sigma \|t_n(v_n - f)\|^2 < \infty$, and $\{v_n\}$ is bounded, then $\{x_n\}$ converges weakly to a point of $E$.

Note, in particular, that if $A$ is bounded on $C$, then for any nonnegative sequence $\{t_n\}$ in $l^2 \setminus l^1$ the conditions on $\{t_n\}$ and $\{v_n\}$ are automatically satisfied.

**Theorem 2.** If $A$ has the property

Iterative Solution of a Variational Inequality

\( \{ z_1 \in Ax_1, z_2 \in Ax_2, (z_1 - z_2, x_1 - x_2) = 0 \} \Rightarrow z_2 \in Ax_1, \)

then \( A \) satisfies (3) on any \( C \) for which \( E \neq \emptyset \).

Condition (4) is satisfied by a very wide class of monotone operators which have arisen in several different contexts: strictly monotone operators, subdifferentials of proper l.s.c. convex functions, the maximal monotone trimonotone operators of Brezis and Browder [1], [2], maximal monotone operators satisfying condition (1) of [1], and the class \( M_2 \) of Browder and Petryshn [5] (single-valued operators satisfying \( (Ax - Ay, x - y) \geq \delta \|Ax - Ay\|^2 \) for some \( \delta > 0 \)). In particular, (2) can be applied to minimize a convex functional on a constraint set under much weaker hypotheses than has heretofore been possible (compare with Goldstein [9]).

Moreover, in an important special case (3) is satisfied by an otherwise arbitrary monotone operator.

Theorem 3. If \( \text{int} \ C \not= \emptyset \), \( C \) is rotund, and (1) has a solution \([u_0, z_0]\) with \( z_0 \neq f \), then \( A \) satisfies (3) and \( u_0 \) is the unique solution of (1). If, in addition, \( C \) is uniformly rotund, then in Theorem 1 the hypothesis that \( \{v_n\} \) is bounded may be deleted and the conclusion strengthened to: \( \{x_n\} \) converges strongly to the solution of (1).

Proofs. In what follows, we shall normalize to \( f = 0 \). For any solution \([u, z]\) of (1) we find

\[
0 \leq 2t_n(z, x_n - u) \leq 2t_n(v_n, x_n - u) \\
\leq ||x_n - u||^2 - ||x_{n+1} - u||^2 + ||t_nv_n||^2
\]

by virtue of the nonexpansiveness of \( P, Pu = u \), the monotonicity of \( A \), and the fact that \([u, z]\) is a solution of (1). Inequality (5) permits three conclusions: (a) \( \lim_n ||x_n - u|| \) exists for each \( u \in E \); (b) \( \Sigma t_n(v_n, x_n - u_0) \) is a convergent positive-term series, hence \( \lim \inf_n (v_n, x_n - u_0) = 0 \); (c) if a subsequence \( \{x_{n(i)}\} \) converges weakly to \( x \) and \( \lim_r(v_{n(i)}, x_{n(i)} - u_0) = 0 \), then \( x \in E \) (because \( 0 \leq (z_0, x_n - u_0) \leq (v_n, x_n - u_0) \) and (3) is satisfied). An appeal to a variant of [3, Lemma 6] (or a direct appeal to Opial's lemma [11]) establishes the uniqueness of \( x \) in (c), i.e.,

\[
\exists x^* \in E \text{ such that } x_{n(i)} \rightharpoonup x^* \forall \{x_{n(i)}\},
\]

satisfying \( \lim_r(v_{n(i)}, x_{n(i)} - u_0) = 0 \).

For \( \delta > 0 \) put \( P = \{n: (v_n, x_n - u_0) \geq \delta\} \) and note that (5), the nonexpansiveness of \( P \), and the boundedness of \( \{v_n\} \) imply that \( \Sigma_{n \in P} \|x_n - x_{n+1}\| \) converges. With (6) this implies \( x_n \rightarrow x^* \).

The proof of Theorem 2 is simple computation which we omit. The proof of Theorem 3 is based on the observation that if \([u_0, z_0]\) is a solution of (1)
with \( z_0 \neq 0 \), then necessarily \( u_0 \in \text{bdry } C \) and the hyperplane \( u_0 + z_0^\perp \) supports \( C \) at \( u_0 \). Any solution \( [x, v] \in A, x \in C \) of \( (u, x - u_0) = 0 \) must have \( x \in u_0 + z_0^\perp \) because \( 0 = (u, x - u_0) \geq (z_0, x - u_0) \geq 0 \); by the rotundity of \( C \), therefore, \( x = u_0 \), i.e., (3) is satisfied. This implies, incidentally, that the solution \( u_0 \) of (1) is unique (although \( z_0 \) may not be unique).

If \( C \) is also uniformly rotund, then, as in the proof of Theorem 1, there exists \( x^* \in E \) for which (6) is valid (and such subsequences \( \{x_{n(t)}\} \) do exist). But a sequence in a uniformly rotund convex set which converges weakly to a point on the boundary must converge strongly; since \( \lim_n \|x_n - x^*\| \) exists, therefore \( \lim_n \|x_n - x^*\| = \lim_n \|x_{n(t)} - x^*\| = 0 \).

For related iterative solutions of \( f \in x + Ax \) and \( f \in Ax \) (without use of the projection \( P \)) see [7], [8]. The proof of Theorem 1 is based on an idea of [6]. Complete proofs, other consequences, extensions of the results of [8] to variational inequalities, as well as sequential analogue of [6, Theorem 5] for even convex functions, will appear elsewhere.

REFERENCES

7. ——, *The iterative solution of the equation \( y \in x + Tx \) for a monotone operator \( T \) in Hilbert space*, Bull. Amer. Math. Soc. 79 (1973), 1258–1261. MR 48 #7034.

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