ON THE NACHBIN TOPOLOGY IN SPACES OF
HOLOMORPHIC FUNCTIONS

BY JORGE MUJICA

Communicated by Hans Weinberger March 27, 1975

1. Introduction. \(H(U)\) denotes the vector space of all holomorphic functions on an open subset \(U\) of a complex Banach space \(E\). In this note we announce results concerning the Nachbin topology \(\tau_\omega\) in \(H(U)\). \(\tau_\omega\) is useful in the study of holomorphic continuation; see Dineen [5], [7] and Matos [8]. We recall the definition of \(\tau_\omega\); see Nachbin [10]. A seminorm \(p\) on \(H(U)\) is said to be ported by a compact subset \(K\) of \(U\) if for each open set \(V\), with \(K \subset V \subset U\), there exists \(c(V) > 0\) such that \(p(f) \leq c(V) \sup_{x \in V} |f(x)|\) for all \(f \in H(U)\). The locally convex topology \(\tau_\omega\) is defined by all such seminorms. To study \((H(U), \tau_\omega)\) we consider the vector spaces of holomorphic germs \(H(K)\) with \(K \subset U\) compact. We endow each \(H(K)\) with the inductive topology given by

\[ H(K) = \lim_{\varepsilon \to 0} H^\infty(K_{\varepsilon}), \]

where \(K_{\varepsilon} = \{x \in E : \text{dist}(x, K) < \varepsilon\}\) and \(H^\infty(K_{\varepsilon})\) denotes the Banach space of all bounded holomorphic functions on \(K\), with the sup norm.

2. Completeness of \((H(U), \tau_\omega)\). The following theorem answers a question raised by Nachbin [11].

**Theorem 1.** \((H(U), \tau_\omega)\) is always complete.

Earlier partial results were given by Dineen [6], Chae [3] and Aron [2] for \(U\) "nice". We give an indication of the proof of Theorem 1. For each compact \(K \subset U\), let \(M^K\) denote the image of the canonical mapping \(H(U) \to H(K)\). After identifying \(H^\infty(K_{\varepsilon})\) with its image in \(H(K)\), we define:

\[ M^K_{\varepsilon} = M^K \cap H^\infty(K_{\varepsilon}), \]

\[ \tilde{M}^K_{\varepsilon} = \text{closure of } M^K_{\varepsilon} \text{ in } H^\infty(K_{\varepsilon}), \]

\[ \mathcal{M}^K = \bigcup_{\varepsilon > 0} \tilde{M}^K_{\varepsilon}. \]

In a diagram we have

__AMS (MOS) subject classifications (1970).__ Primary 46E10, 46E25; Secondary 32D10.

__Key words and phrases._ Nachbin topology, holomorphic germ, inductive limit, projective limit, multiplicatively locally convex algebra, spectrum, envelope of holomorphy.

\(^1\) The results in §2 of this note are taken from the author's doctoral dissertation at the University of Rochester, written under the supervision of Professor Leopoldo Nachbin.

Copyright © 1975, American Mathematical Society
We endow $M^K$ and $\tilde{M}^K$ with the inductive topologies coming from

$$M^K = \lim_{\epsilon > 0} M^K_\epsilon, \quad \tilde{M}^K = \lim_{\epsilon > 0} \tilde{M}^K_\epsilon.$$  

Theorem 1 follows from Lemmas 1 and 2, below.

**LEMMA 1.** $\tilde{M}^K$ is the completion of $M^K$.  

**LEMMA 2.** $(H(U), \tau_\omega) = \lim_{K \subseteq U} M^K = \lim_{K \subseteq U} \tilde{M}^K$.  

3. **Multiplicative local convexity of** $(H(U), \tau_\omega)$. The following theorem answers a question raised by Matos [8].

**THEOREM 2.** $(H(U), \tau_\omega)$ is a multiplicatively locally convex algebra, i.e. $\tau_\omega$ is defined by the continuous seminorms $p$ such that, for all $f, g \in H(U)$,

$$p(fg) \leq p(f) \cdot p(g).$$

With the notation of §2 we have

**LEMMA 3.** $M^K$ is a multiplicatively locally convex algebra.

Theorem 2 follows from Lemma 2 and Lemma 3.

**REMARK.** The spectrum of the multiplicatively locally convex algebra $(H(U), \tau_\omega)$ can be used to give a construction of the envelope of holomorphy of $U$; see Matos [8]. For similar constructions with other devices see Alexander [1], Coeuré [4] and Schottenloher [12].

**REFERENCES**

10. L. Nachbin, *Topology on spaces of holomorphic mappings*, Ergebnisse der Math-

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627

Current address: Instituto de Matemática, Universidad Católica de Chile, Casilla 114-D, Santiago, Chile