CONVEXITY FOR A SIMPLY CONNECTED \( p \)-ADIC GROUP

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In [4] Kostant showed that the set of Iwasawa double cosets which intersect a given Cartan double coset in a semisimple Lie group corresponds to a certain convex subset in the Lie algebra of a maximal torus of the group. As a consequence, he established that representatives for the double cosets relative to a maximal compact subgroup of a semisimple Lie group may be chosen in the unipotent radical of a minimal parabolic subgroup. We announce here analogues of these results for a simply-connected \( p \)-adic group.

I wish to thank Roger Howe and Jacques Tits for valuable conversations during the time I was thinking about this problem.

Let \( G \) be a connected simply-connected semisimple algebraic group defined over a \( p \)-adic field \( \Omega \). Let \( \mathfrak{g} \) denote the group of \( \Omega \)-rational points of \( G \). Then \( \mathfrak{g} \) is a locally compact totally disconnected group. Borel and Tits [1] and Bruhat and Tits [2] have shown that \( G \) has a structure theory which is similar in many ways to that of a semisimple Lie group.

Let \( P \) be a minimal parabolic \( \Omega \)-subgroup of \( G \). Then \( P \) is a split product \( P = MN \) in which \( M \) is connected and reductive and \( N \) is the unipotent radical of \( P \). Let \( A \) be the maximal \( \Omega \)-split torus in the center of \( M \). For simplicity, we denote the group of \( \Omega \)-points of each of the above algebraic groups by the corresponding ordinary capital letter.

Let \( ^{0}A \) be the maximal compact subgroup of \( A \). Then \( A/^{0}A \) is a free \( \mathbb{Z} \)-module of rank equal to the \( \Omega \)-rank of \( G \). We call \( \mathfrak{a} = (A/^{0}A) \otimes_{\mathbb{Z}} \mathbb{R} \) the Lie algebra of \( A \) and write \( H: A \to \mathfrak{a} \) for the natural map which imbeds \( A/^{0}A \) as a lattice in \( \mathfrak{a} \). To any rational character \( \chi \) of \( A \) we associate a linear functional on \( \mathfrak{a} \) by setting \( \log |\chi(a)| = \langle \chi, H(a) \rangle \) \((a \in A)\). The relative Weyl group \( W = \text{N}_G(A)/\text{Z}_G(A) \) operates on \( A \) and \( \mathfrak{a} \). There is a root system in the dual \( \mathfrak{a}^* \) of \( \mathfrak{a} \) associated to the restriction to \( A \) of the adjoint representation of \( G \). Choosing a \( W \)-invariant scalar product on \( \mathfrak{a} \), we regard this root system as a subset of \( \mathfrak{a} \). Let \( N \) correspond to a set of positive roots and let \( \bar{N} \) be the \( \Omega \)-points of the unipotent radical of the opposite parabolic subgroup, corresponding to the negative roots. Write \( \mathfrak{a}^+ \) [respectively, \( + \mathfrak{a} \)] to denote the (closed) positive chamber in \( \mathfrak{a} \) [respectively, the cone consisting of nonnegative linear combinations of the positive roots]. The mapping \( H \) extends to \( M \); denoting the kernel of \( H \) in \( M \) as \( \mathfrak{o}M \), we see that \( H \) also imbeds \( M/\mathfrak{o}M \) as a lattice in \( \mathfrak{a} \). Let \( M^+ = H^{-1}(\mathfrak{a}^+) \) and \( +M = H^{-1}(+\mathfrak{a}) \).

We choose a particularly "good" maximal compact subgroup \( K \) of \( G \) (i.e. 


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corresponding to the origin in an apartment of the building of $G$ and such that $K \supset B \supset P \cap K = P \cap B$, $B$ a certain Iwahori subgroup of $G$), the existence of which is guaranteed by Bruhat's and Tits' theory, and recall that $G$ has, with respect to $K$, both a Cartan decomposition $G = KM^+K$ and an Iwasawa decomposition $G = NMK$. We have the bijections $K \backslash G / K \leftrightarrow M^+ / \mathfrak{o}M$ and $N \backslash G / K \leftrightarrow M / \mathfrak{o}M$.

For $S \subset \mathfrak{H}$, we write $C(S)$ to denote the convex closure of $S$ and $W \cdot S$ to denote the Weyl group orbit of $S$. Our main result (cf. Theorem 4.1 of [4]) is

**Theorem.** Let $m, m' \in M$. Then the double cosets $KmK$ and $Nm'K$ intersect if and only if $H(m') \in C(W \cdot H(m))$.

Since $0 = H(1) \in C(W \cdot x)$ for all $x \in \mathfrak{a}$, we have

**Corollary.** $G = KNK$.

The following simple geometric lemma translates "only if" into a result [2, 4.4.4(iii)] of Bruhat-Tits.

**Lemma 1.** Let $x \in \mathfrak{a}^+$ and set $\mathfrak{a}(x) = \{ y \in \mathfrak{a}^+ | x - y \in + \mathfrak{a} \}$. Then $W \cdot \mathfrak{a}(x) = C(W \cdot x)$.

It follows by inspection of the defining integrals that the zonal spherical functions on $G$, evaluated at fixed $m \in M$ and regarded as functions on $\mathfrak{a}$ (or rather on $\mathfrak{a}^*$), may be interpreted as generating functions for the measures of the intersections $KmK \cap Nm'K$ ($m' \in M$). The Weyl group invariance of such a function on $\mathfrak{a}^*$ [5] or, what is the same, the equivalence of all principal series representations of $G$ in a Weyl group orbit [3, §8] implies

**Lemma 2.** Let $m, m' \in M$. Then $KmK$ intersects $Nm'K$ for $N = N'$ if and only if the same is true for some (i.e. for every) $N'$ conjugate to $N$ under $W$.

The "if" part of our theorem is an easy consequence of the following structural result.

**Lemma 3.** Let $B$ and $\mathcal{N}$ be as above. Then $B\mathcal{N}K = B + MK$.

Details and proofs will appear elsewhere.

**References**


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