ERGODIC EQUIVALENCE RELATIONS,  
COHOMOLOGY, AND VON NEUMANN ALGEBRAS  
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1. Introduction. Throughout, $(X, \mathcal{B})$ will be a standard Borel space, $G$ some countable group of automorphisms, $R_G$ the equivalence relation $\{(x, g \cdot x), g \in G\}$, and $\mu$ a $\sigma$-finite measure on $X$. For $\mu$ quasi-invariant, the orbit structure of the action has been studied by Dye [4], [5], Krieger [8]–[13], and others. Here, ignoring $G$ and focusing on $R_G$ via an axiomatization, and studying a cohomology for $R_G$, we obtain a variety of results about group actions and von Neumann algebras. The major results are stated below.

2. Equivalence relations. $R$ will be an equivalence relation on $X$ with all equivalence classes countable, and $R \in \mathcal{B} \times \mathcal{B}$.

**Theorem 1.** Every $R$ is an $R_G$.

Properties of $G$-actions translate into properties of $R_G$ which can be stated with no $G$ in sight, e.g., quasi-invariance, ergodicity. Let $\mu$ be quasi-invariant, and let $C = \mathcal{B} \times \mathcal{B}|_R$ and $P_t(x, y) = x$, $P_r(x, y) = y$. Now $C$ has a natural measure class as follows:

**Theorem 2.** The formula $v_\mu(C) = \int P_t^{-1}(x) \cap C d\mu(x)$, where $|\cdot|$ is cardinality, and a similar formula for $v_r$ define equivalent $\sigma$-finite measures on $C$.

The Radon-Nikodym derivative is the function $D = dv_r/dv_\mu$ on $R$. If $R = R_G$, then $d(\mu \cdot g)/d\mu(x) = D(x, gx)$. Moreover, $D$ is a cocycle in that $D(x, y)D(y, z) = D(x, z)$ a.e. and the $D'$ arising from a $\mu'$ equivalent to $\mu$ is cohomologous to $D$.

For ergodic $R$, one has a classification into types which are $I_n$, $n = 1, \ldots, \infty$, $II_1, II_\infty$ and III as in [3]. For $j = 1, 2$, relations $R_j$ on $(X_j, \mathcal{B}_j, \mu_j)$ are isomorphic if there is a Borel isomorphism $a: X_1 \to X_2$ with $\mu \sim \mu \circ a^{-1}$ and $R_2(a(x)) = a(R_1(x))$ a.e. If the $R_j$ are ergodic, they are principal groupoids and, hence, define virtual groups [14].

**Theorem 3.** $R_1$ and $R_2$ define isomorphic virtual groups iff each is isomorphic to a restriction of the other, where the restriction of $R$ to $H$ is $R \cap H \times H$. Hence, the two notions of isomorphism coincide if $R_1$ and $R_2$ are both of infinite type.

Hyperfiniteness in terms of $R$ becomes: $\exists R_n \uparrow R$ with $|R_n(x)|$ finite $\forall n, \forall x$. 


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3. Cohomology. For simplicity assume that $R$ is ergodic and let $R^n = \{(x_0, \ldots, x_n), x_0 \sim \cdots \sim x_n\} \subset X^{n+1}$ with the natural measure class generalizing that of Theorem 2. An $R$ module is an abelian polonais group with a Borel map $u$ of $R$ into $\text{Aut}(A)$ with $u(x, y)u(y, z) = u(x, z)$. Define cochain groups $C^n(R, A)$ as the Borel functions mod null functions from $R^n$ to $A$ with coboundary operators $(d^nc)(x_0, \ldots, x_{n+1}) = \Sigma j(-1)^jc(x_0, \ldots, \hat{x}_j, \ldots, x_n)$ if $u = 1$ and with a slight modification if $u \neq 1$. We define cohomology groups $H^n(G, A)$ of this complex; also for $n = 1$, we allow $A$ to be nonabelian and obtain a cohomology set. These groups were introduced in the virtual group context by Westman [17]. We show how to axiomatize these groups and show that they are unique solutions to a universal problem. If $R \sim R_G$ with $G$ acting freely, one may identify $H^n(R, A)$ with $H^n(G, U(X, A))$, where $U(X, A)$ is Borel functions mod null functions from $X$ to $A$ with $G$ operating suitably. If $R$ is hyperfinite and not type $I_n$ so that $R = R_Z$, with $Z$ acting freely, then $H^n(R, A) = 0$ for all $n > 2$.

Since any action of an abelian group is hyperfinite (Dye [5], Feldman and Lind [6]), one can obtain results of the following kind:

**Theorem 4.** If $s$ and $t$ are commuting ergodic independent $(s^n \neq t^m)$ automorphisms of $(X, \mu)$, then for any Borel function $f$ from $X$ to the circle $T$, there exist Borel functions $g$ and $h$ to $T$ so that $f = ((g \circ s)/g)((h \circ t)/h)$ a.e.

Generalizing Mackey [14], we define for $c \in Z^1(R, A)$ a relation $R(c)$ on $X \times A$ by $(x, a) \sim (x^1, a^1)$ iff $x \sim x^1$ and $c(x, x^1)a = a^1$, where $A$ is an abelian locally compact $R$ module with trivial action. Then $A$ acts by right translations on $X \times A$ and preserves $R(c)$ and so acts via Mackey's point realization theorem on $Z = X \times A/R(c)$, where $\widehat{R(c)}$ is a countably separated equivalence relation containing $R(c)$ whose image in the measure algebra of $X \times A$ coincides with the $R(c)$ invariant sets. This ergodic action of $A$ is called the range of $c$, and depends only on the class of $c$. The isotropy group $A_z$ of $A$ at $z \in Z$ is an a.e. constant closed subgroup $A(c)$ which is called the proper range of $c$. Now if $A^*$ is the one point compactification of $A$, we generalize [12] and define the asymptotic range $r^*(c)$ as the intersection over all subsets $B$ of $X$ of positive measure of the essential ranges in $A^*$ of $c$ restricted to $B \times B$, and $r^*(c) = r^*_\infty(c) \cap A$. An important result is

**Theorem 5.** $r^*(c)$ is a closed subgroup of $A$ depending only on the class of $c$ and equals the proper range $A(c)$ of $c$.

For $A = R$ and $c = \log D$, this was done by enumeration of cases in [7], and there is some overlap with results in [2]. We also have

**Theorem 6.** For $A = R^n + Z^m, c \sim 0$ iff $\in \notin r^*_\infty(c)$.

As a corollary we obtain the result that if $\log D$ is bounded, then there is an equivalent invariant measure, a result that also follows from Theorem 1 and [15].
4. von Neumann algebras. Generalizing the Zeller-Meier generalization [18] of the Murray-von Neumann group measure space factors, we construct for an ergodic relation $R$ and $t \in H^2(R, T)$ a factor $M(R, t)$ which we view as the "twisted algebra of matrices over $R$". For $t = 1$ this factor is constructed (less transparently) in [10]. Our Hilbert space $H$ is $L_2(R, \nu_1)$, and we pick $c \in t$ normalized to be skew symmetric. For $F \in L^\infty(R)$ which is band limited in that $\{x|F(x, y) \neq 0$ and $0 \neq F(y, x)\}$ is bounded, one defines an operator $M_F$ on $H$ by

$$(M_F f)(x, z) = \sum_{y \sim x} f(x, y)F(y, z)c(x, y, z).$$

These operators form a *-algebra whose weak closure is a factor $M(R, t)$ depending only on $t$ and not on $c$. The commutant has a similar form. The indicator function of the diagonal $A$ is a separating and cyclic vector, and the diagonal subalgebra $A = \{M_F, F = 0 \text{ off } \Delta\}$ is a maximal abelian subalgebra which is regular by Theorem 1. Moreover, there is a normal faithful conditional expectation $E$ of $M(R, t)$ onto $A$. If $M$ is any factor with abelian subalgebra $A$ satisfying these conditions, we call $A$ a Cartan subalgebra [19]. One of our major results is a converse of this construction.

**Theorem 7.** If $A$ is a Cartan subalgebra of the factor $M$, then $M = M(R, t)$ for suitable $R$ and $t$ with $A$ as diagonal subalgebra for any $R'$. Of course, if $M$ is a finite factor, the $E$ always exists. One may ask if $M(R, t)$ determines $R$ and $t$. If we restrict to hyperfinite $R$ (where $t = 1$ automatically), then $M(R, 1)$ does indeed determine $R$ by [4], [5], [6]. A major open problem is whether we get all factors as $M(R, t)$'s. We note that Connes [1] constructs an $M(R, t)$ which is not an $M(R', 1)$.

Our final results concern automorphisms and conjugacy questions. If $A$ is the diagonal subalgebra of $M = M(R, t)$, let $Out(M, A)$ be the subgroup of the "outer" automorphism group of $M$ which maps $A$ into something inner conjugate to $A$. Let $Out(R, t)$ be the group of "outer" automorphisms of the relation $R$ fixing the cohomology class $t$. We have a structure theorem for $Out(M, A)$ generalizing results in [16].

**Theorem 8.** We have an exact sequence $1 \to H^1(R, T) \to Out(M, A) \to Out(R, t) \to 1$.

Finally, let $A_i$ be two Cartan subalgebras of $M$ with conditional expectations $E_i$. The restriction of $E_1$ to $A_2$ gives rise to a unique positive measure $\gamma$ on $X_1 \times X_2$ (where $A_i = L^\infty(X_i, \mu_i)$) whose disintegration products $\gamma_x$ ($x \in X_1$) with respect to projection to $X_1$ are determined by $E_1(a)(x) = \int a(y) d\gamma_x(y)$ a.e. for $a \in A_2 = L^\infty(X_2, \mu_2)$. Let us say that $A_2$ is discrete over $A_1$ if a.a. $\gamma_x$ are atomic measures.

**Theorem 9.** If $M$ is an infinite factor, $A_1$ and $A_2$ are inner conjugate iff $A_2$ is discrete over $A_1$ and $A_1$ is discrete over $A_2$. 
ADDED IN PROOF. Theorem 5, for hyperfinite $R$, was also obtained by K. Schmidt (Cohomology and skew products of ergodic transformations, University of Warwick, Coventry, England, preprint).

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