ON THE CLASSIFICATION OF TAUT SUBMANIFOLDS

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All terminology will be smooth. A submanifold $K^{2n} \hookrightarrow M^{2n+2}$ is taut if $\pi_i(U, \partial U) = 0$ for $i < n$, where $U = (M$-neighborhood $K$). Examples are: nonsingular algebraic hypersurfaces in $CP^n$ (this follows from the Lefschetz theorem on hyperplane sections), simple knots (see [L]), the spines (see [M]). Every codimension-2 homology class contains taut representatives (see [K-M]), and the set of taut submanifolds is closed under connected sum (of pairs) with $(S^n \times S^n \rightarrow S^{2n+2})$. Taut submanifolds are "almost canonical" in the sense of [Q], and from this viewpoint it is readily seen that if $n \geq 3$, every $K^{2n} \hookrightarrow M^{2n+2}$ with $i$-n-connected is concordant to $K^{2n} \hookrightarrow M^{2n+2}$ taut.

If $M^{2n+2}$ is simply connected, the homology groups of $K^{2n}$, taut, are completely determined by the homology of $M^{2n+2}$ except for $B_n(K^{2n})$. A lower bound on $B_n(K)$ in terms of $i_*[K^{2n}]$ and the cohomology ring of $M^{2n+2}$ has been obtained in [T-W]. In [F1] we have proven Theorem 1, which provides a partial converse to Theorem 2.2 of [T-W] for $M \cong CP^{n+1}$, $n > 2$ odd, and $i_*[K] = p$, a prime, multiple of the generator of $H_{2n}(CP^{n+1}; Z)$. Interestingly, if $p > 3$, the nonsingular algebraic hypersurfaces $V$ are not the simplest taut submanifolds in their homology class, but may be decomposed as $V = K \# l$-copies $S^n \times S^n$, $l > 0$, for some taut submanifold $K$.

We do not know if this is true for $n = 1$. If it were, there would be surfaces imbedded in $CP^2$ with genus smaller than that of the nonsingular algebraic hypersurfaces to which they are homologous. This would contradict Thom's conjecture.

Statement of Theorem 1. Let $M^{2n+2}$ be a simply-connected, oriented, smooth $(2n+2)$-manifold, $n$ odd $> 1$. Let $x \in H^2(M^{2n+2}; Z)$ generate a free summand of $H^2(M^{2n+2}; Z)$. Let $p$ be any prime. Set

$$ s_{\text{even}} = \max \{4, (\cosh(p - 2k)x)(\text{sech}(px))(L(M))[M] | 0 < k < p \}, $$
$$ s_{\text{odd}} = \max \{3, (\cosh(p - 2k)x)(\text{sech}(px))(L(M))[M] | 0 < k < p \}, $$

where $L$ is the Hirzebruch polynomial.

For all integers $h \geq 0$, there exists a taut submanifold $K_h \hookrightarrow M$ with

$$ M \cap px = i_*[K_h], $$

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and

\[ B_n(K_h) = \overline{\alpha}_{\text{even}} + 6\tau_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \]

if \( B_{n+1}(M) \) is even.

\[ = \overline{\alpha}_{\text{odd}} + 6\tau_n(M) - 2B_n(M) + B_{n+1}(M) + 2h, \]

if \( B_{n+1}(M) \) is odd,

\[ B_n(M) = \text{rank } H_n(M; Z)/\text{Torsion}, \quad T_n(M) = \text{rank } H_n(M) = \text{rank } H_n(M; Z). \]

We now state two theorems, proved in [F2], which indicate to what extent the diffeomorphism class of a taut submanifold is fixed by \( B_n(K) \).

**Theorem 2.** If \( M^{2n+2} \) is a compact, simply connected, smooth \((2n+2)\)-manifold, \( n \) odd \( \geq 3 \), and \( K_0^{2n+2} \xrightarrow{i_0} M^{2n+2} \) and \( K_1^{2n+2} \xrightarrow{i_1} M^{2n+2} \) are \( n \)-connected inclusions of closed submanifolds with \((i_0)_*[K_0] = (i_1)_*[K_1] \in H_{2n}(M^{2n+2}; Z)\), then if \( B_n(K_0) = B_n(K_1) \), \( K_0 \) is diffeomorphic to \( K_1 \).

**Theorem 3.** Assume \( M^{2n+2} \) is a simply-connected smooth \((2n+2)\)-manifold, \( n \) even, \( \geq 2 \), with \( H_n(M; Z) = 0 \). If \( i_0 \) and \( i_1 \) are as above, then if the intersection pairings on \( H_n(K_0; Z)/\text{Torsion} \) and \( H_n(K_1; Z)/\text{Torsion} \) are isometric, \( K_0 \) is diffeomorphic to \( K_1 \).

If \( M^{2n+2} \), \( n \) odd, \( \geq 3 \), is simply-connected, it follows from Theorem 2 that there is a simplest taut submanifold representing \( i_*[K] \), \( K_0 \), and every other is of the form \( K_i = K_0 \#_l \text{copies } S^n \times S^n \). This, together with a previous remark, yields a complete classification of taut submanifolds in a homotopy \( CP^{n+1}, n \) odd, \( \geq 1 \), representing a prime multiple of the generator of \( H_{2n}(CP^{n+1}; Z) \).

**References**


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