DUALITY FOR CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS BY LOCALLY COMPACT GROUPS

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The duality for crossed products of von Neumann algebras by locally compact abelian groups has been obtained by Takesaki [4]. We shall generalize this result to a locally compact (not necessarily abelian) group by using the Fourier algebra in place of the dual group.

Let $G$ denote a locally compact group with a right invariant Haar measure $dt$, and $M$ denote a von Neumann algebra over a Hilbert space $\mathcal{H}$. By an action of $G$ on $M$ we mean a homomorphism $\sigma: t \in G \mapsto \sigma_t \in \text{Aut}(M)$ such that for each $x \in M$ the mapping $t \in G \mapsto \sigma_t(x)$ is $\sigma$-strongly* continuous. Let $\{\pi_\sigma, \lambda\}$ be a covariant representation of $\{M, \sigma\}$ on $\mathcal{H} \otimes L^2(G)$ defined by

\[
\begin{align*}
(\pi_\sigma(x)\xi)(s) & \equiv \pi_s(x)\xi(s), & \xi & \in \mathcal{H} \otimes L^2(G), \\
\lambda(r)\xi(s) & \equiv \xi(sr), & r, s & \in G.
\end{align*}
\]

Then the crossed product $R(M; \pi_\sigma)$ of $M$ by $G$ is the von Neumann algebra generated by $\pi_\sigma(M)$ and $\lambda(G)$.

**THEOREM 1.** A necessary and sufficient condition that a mapping $\alpha$ of $M$ into $M \otimes L^\infty(G)$ be induced by an action $\sigma$ with

\[
(\alpha(x)\xi)(s) = \sigma_s(x)\xi(s), \quad x \in M, \xi \in \mathcal{H} \otimes L^2(G),
\]

is that $\alpha$ be an isomorphism with the commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & M \otimes L^\infty(G) \\
\downarrow{\alpha} & & \downarrow{\alpha \otimes \iota} \\
M \otimes L^\infty(G) & \xrightarrow{\iota \otimes \delta} & M \otimes L^\infty(G) \otimes L^\infty(G),
\end{array}
\]

where $(\delta f)(s, t) \equiv f(st)$ for $f \in L^\infty(G)$.

For the right regular representation $\lambda_G$ of $G$ on $L^2(G)$, i.e.,

\[
(\lambda_G(s)f)(t) \equiv f(ts), \quad f \in L^2(G), s, t \in G,
\]

let $R(G)$ denote the von Neumann algebra generated by $\lambda_G(G)$. Let $\gamma$ denote the isomorphism of $R(G)$ into $R(G) \otimes R(G)$ defined by

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DEFINITION. For an isomorphism $\beta$ of a von Neumann algebra $N$ into $N \otimes R(G)$ with the commutative diagram:

\[
\begin{array}{ccc}
N & \xrightarrow{\beta} & N \otimes R(G) \\
\beta & & \beta \otimes \iota \\
N \otimes R(G) & \xrightarrow{\iota \otimes \gamma} & N \otimes R(G) \otimes R(G),
\end{array}
\]

we define a crossed dual product of $N$ by $G$ as the von Neumann algebra generated by $\rho(N)$ and $1 \otimes L^\infty(G)$. We denote it by $R_d(N; \beta)$.

\[\gamma(\lambda_G(s)) \equiv \lambda_G(s) \otimes \lambda_G(s), \quad s \in G.\]

THEOREM 2. Let $W$ and $V$ be unitaries on $H \otimes L^2(G) \otimes L^2(G)$ defined by

\[(W\xi)(s, t) \equiv \xi(s, ts) \quad \text{and} \quad (V\xi)(s, t) \equiv \xi(st, t).\]

If $\alpha$ (resp. $\beta$) is an isomorphism of $M$ (resp. $N$) into $M \otimes L^\infty(G)$ (resp. $N \otimes R(G)$) with the commutative diagram (1) (resp. (2)), then $\hat{\alpha}$ (resp. $\hat{\beta}$) defined by

\[\hat{\alpha}(y) \equiv W^*(y \otimes 1)W \quad \text{(resp.} \quad \hat{\beta}(z) \equiv V(z \otimes 1)V^*)\]

is an isomorphism of $R(M; \alpha)$ (resp. $R_d(N; \beta)$) into $R(M; \alpha) \otimes R(G)$ (resp. $R_d(N; \beta) \otimes L^\infty(G)$) with the commutative diagram (2) for $R(M; \alpha)$ and $\hat{\alpha}$ (resp. (1) for $R_d(N; \beta)$ and $\hat{\beta}$).

Making use of the above two theorems we can give the following duality theorem for crossed products of von Neumann algebras by locally compact groups. When $G$ is abelian, its corollary is nothing but a duality theorem of Takesaki [4].

THEOREM 3 (DUALITY). Under the notations in Theorem 2, let $\sigma$ be an action of $G$ on $M$, $\alpha \equiv \pi$, $\beta \equiv \hat{\alpha}$, $\tilde{\alpha} \equiv \hat{\beta}$ and $\tilde{\sigma}$ the action associated with $\tilde{\alpha}$ as in Theorem 1. Let $\pi$ be a faithful representation of $M$ on $H \otimes L^2(G) \otimes L^2(G)$ such that

\[(\pi(x)\xi)(s, t) = \sigma_{st^{-1}}(x)\xi(s, t),\]

and let $\Lambda_1$ and $\Lambda_2$ be a representation and a unitary representation of $G$ on $H \otimes L^2(G) \otimes L^2(G)$ defined by

\[(\Lambda_1(r)\xi)(s, t) \equiv \xi(s, r^{-1}t) \quad \text{and} \quad (\Lambda_2(r)\xi)(s, t) \equiv \xi(s, tr),\]

respectively. Then $R_d(R(M; \alpha); \beta)$ is isomorphic to $\pi(M) \otimes B(L^2(G))$ and the isomorphism transforms the action $\sigma$ of $G$ on the former into the action of $G$ on the latter given by $\Ad(\Lambda_2(r)) \otimes \Ad(\lambda_G(r))$ for $r \in G$. In particular,

\[\pi(\sigma_r(x)) = \Lambda_1(r)\pi(x)\Lambda_1(r)^{-1} \quad \text{and} \quad \tilde{\gamma}(\pi(x)) = \Lambda_2(r)\pi(x)\Lambda_2(r)^{-1}.\]

When $G$ is unimodular, we can define a unitary $U$ on $H \otimes L^2(G) \otimes L^2(G)$ by

\[(U\xi)(s, t) \equiv \Delta(t)^{\frac{1}{2}}\xi(t^{-1}s, t),\]
and a mapping $\hat{\beta}'$ of $\mathcal{R}_d(N; \beta)$ into $\mathcal{R}_d(N; \beta) \otimes L^\infty(G)$ by

$$\hat{\beta}'(z) \equiv U(z \otimes 1)U^*.$$

Then $\hat{\beta}'$ is an isomorphism which makes commutative the diagram (1) for $\mathcal{R}_d(N; \beta)$ and $\hat{\beta}'$.

**COROLLARY.** Assume that $G$ is unimodular. Under the notations in Theorem 2, let $\sigma$ be an action of $G$ on $M$, $\alpha \equiv \pi_\sigma$, $\beta \equiv \hat{\alpha}$, $\bar{\sigma} \equiv \hat{\beta}'$ and $\bar{\sigma}$ the action associated with $\bar{\alpha}$ as in Theorem 1. Then $\mathcal{R}_d(\mathcal{R}(M; \alpha); \beta)$ is isomorphic to $M \otimes B(L^2(G))$ and the isomorphism transforms the action $\bar{\sigma}$ of $G$ on the former into the action of $G$ on the latter given by $\sigma_r \otimes \text{Ad}(\lambda^G_r)$ for $r \in G$, where $\lambda^G_r$ is the left regular representation of $G$ on $L^2(G)$.

**THEOREM 4 (DUALITY).** Under the notations in Theorem 2, $\mathcal{R}(\mathcal{R}_d(N; \beta); \alpha)$ is isomorphic to $N \otimes B(L^2(G))$.

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**REFERENCES**


2. A. Ikunishi and Y. Nakagami, *On invariants $G(\sigma)$ and $I(\sigma)$ for an automorphism group of a von Neumann algebra*, Publ. RIMS, Kyoto, Univ. (to appear).
