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CONSTRUCTION OF SOME NEW FOUR-DIMENSIONAL MANIFOLDS

BY SYLVAIN E. CAPPELL AND JULIUS L. SHANESON[1]

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This note announces a new construction in the theory of 4-manifolds.

Let $\varphi\colon T^3 \longrightarrow T^3$, $T^3 = S^1 \times S^1 \times S^1$, the torus of dimension three, be a diffeomorphism, with $\varphi(x) = x$, some $x \in T^3$. Let A be a matrix for the map φ induces on $\pi_1 T^3 = Z \oplus Z \oplus Z$. Assume that $\det A = -1$ and $\det(I - A^2) = \pm 1$, I = identity matrix. It is easy to see that such a map φ exists.

Let the manifold M be obtained from $T^3 \times [0, 1]$ by identifying $(y, 0)$ with $(\varphi(y), 1)$. Let M_0 be the complement of the interior of a tubular neighborhood of the image of $\{x\} \times [0, 1]$ in the quotient M. Clearly ∂M_0 can be identified with the boundary $S(\rho)$ of the nontrivial disk bundle $D(\rho)$ over S^1 with group $O(4)$. There is a (canonical) map $h\colon M_0 \longrightarrow D(\rho)$ restricting to the identity on $S(\rho)$.

Let N be any connected nonorientable 4-manifold, and let N_0 be the complement of the interior of a tubular neighborhood of a circle in N representing an element $\alpha \in \pi_1 N$ that reverses orientation. Then $\partial N_0 = S(\rho)$. Let

$$Q_N = Q_{N,A} = N_0 \cup_{S(\rho)} M_0$$

and let $h_N = \mathrm{id}_{N_0} \cup h_0$; i.e., Q_N is obtained from the disjoint union of N_0 and M_0 by identifying their boundaries.

THEOREM. *Suppose α has order two. Then*

(i) *h_N is a simple homotopy equivalence,*

(ii) *h_N is not homotopic to a diffeomorphism (or even to a PL homeomorphism).*

For example, let N be real projective 4-space. Then Q_N is not diffeomorphic or even PL homeomorphic or PL s-cobordant to N. In fact, there are exactly two s-cobordism classes of homotopy 4-dimensional real projective spaces. In particular one has

THEOREM. *There is a smooth free action of the group of order two, on a homotopy 4-sphere, that is not equivariantly diffeomorphic (or even PL homeomorphic) to a linear action on the standard 4-sphere.*

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, NEW YORK, NEW YORK 10012 (Current address of S. E. Cappell)

Current address (J. L. Shaneson): Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903