BOUNDARY VALUE PROBLEMS FOR
EVEN ORDER NONLINEAR ORDINARY
DIFFERENTIAL EQUATIONS

BY R. KANNAN AND J. SCHUUR

Communicated by Richard K. Miller, September 15, 1975

We outline an operator-theoretic method for proving the existence of solutions of boundary value problems where the nonlinearity satisfies a Nagumo condition. In this note we explain the ideas by considering a particular fourth order problem. The method involves (i) converting the boundary value problem to an "alternative problem" [2]; (ii) inducing a splitting of the integral operator in the alternative problem by splitting the original differential operator [3]; and (iii) applying degree theory.

Let \( S \) be the Hilbert space \( L^2[0, 1] \) with the inner product and norm denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) respectively. Let \( H^2[0, 1] \) be the Hilbert space of all functions \( x(t) \) such that \( x(t) \) and its first \((n - 1)\) derivatives are absolutely continuous and \( x^{(n)}(t) \in L^2[0, 1] \).

**Theorem.** Let \( f(t, x, y, z) \) be continuous from \([0, 1] \times R \times R \times R \) into \( R \) such that

(i) \( |f(t, x, y, z)| \leq a + b|x| + c|y| + Q(|z|) \) where \( Q: [0, \infty) \rightarrow R \) is a positive continuous nondecreasing function satisfying \( \lim_{s \rightarrow \infty} Q(s)/s^2 < \infty \); and

(ii) there exists \( R_1 > 0 \) such that \( \|x\| = R_1, x \in H^2[0, 1] \) implies \( \langle x, f(t, x, x', x'') \rangle + \|x''\|^2 \geq 0 \).

Then the nonlinear boundary value problem

\[
(1) \quad x'''' + f(t, x, x', x'') = 0, \quad x'(0) = x'(1) = x'''(0) = x'''(1) = 0
\]

has at least one solution.

**Outline of Proof.** Let \( L \) and \( T \) be the linear operators defined by

\[
\mathcal{D}(L) = \{x \in H^4[0, 1] : x'(0) = x'(1) = x'''(0) = x'''(1) = 0\}, \quad Lx = x''''
\]

\[
\mathcal{D}(T) = \{x \in H^2[0, 1] : x'(0) = x'(1) = 0\}, \quad Tx = x''
\]

Then, if \( T^* \) denotes the adjoint of \( T \), it can be seen that \( T = T^* \) and \( L = TT^* \).


**Key words and phrases.** Periodic boundary value problems, alternative problems, Leray-Schauder degree, Nagumo condition.
Let the nonlinear Nemitsky operator $N$ be defined by
$$
D(N) = H^2[0, 1], \quad (Nx)(t) = f(t, x, x', x'').
$$

We can rewrite (1) as
$$
Lx + Nx = 0.
$$

Let $S_0$ be the null-space of $L$ and $S_1 = S_0^\perp$. Let $P: S \to S_0$ be the projection operator. By applying the method of "alternative problems," we can transform (2) (cf. [2]) into

(3) 
$$
x + H(I - P)Nx = x^*,
$$

(4) 
$$
PNx = 0
$$

where $x^* \in S_0$ and $H = [L|D(L) \cap S_1]^{-1}$.

The decomposition of $L$ into $TT^*$ induces a decomposition of $H(I - P)$ into $J^*J$ where $J^*$ and $J$ are the appropriate inverses of $T^*$ and $T$, respectively. If $x_1 \in S_1$ is such that $x = J^*x_1 + x^*$ then (3) and (4) can be written as [3]

(5) 
$$
x_1 + JN(J^*x_1 + x^*) = 0,
$$

(6) 
$$
PN(J^*x_1 + x^*) = 0.
$$

Solving this system of equations is equivalent to solving

(7) 
$$
(I - R)z = 0
$$

where $z = (x_1, x^*) \in S_1 \oplus S_0$ and $R: S_1 \oplus S_0 \to S_1 \oplus S_0$ is defined by
$$
Rz = (- JN[J^*x_1 + x^*], x^* - PN[J^*x_1 + x^*]).
$$

Now $J^*$ is continuous from $S_1$ into $H^2[0, 1]$; $N$ maps bounded sets of $H^2[0, 1]$ continuously into bounded subsets of $L^1[0, 1]$, by virtue of the Nagumo condition; and $J$ maps these bounded sets into compact sets in $S_1$. Hence, $R$ is a compact map of $S_1 \oplus S_0$ into itself. Also,

$$
\langle (I - R)z, z \rangle_{S_1 \oplus S_0} = \|x_1\|^2 + \|x^*\|^2 + \langle x_1, JN[J^*x_1 + x^*] \rangle
\quad - \langle x^*, PN[J^*x_1 + x^*] \rangle
\quad = \|x_1\|^2 + \langle J^*x_1 + x^*, N[J^*x_1 + x^*] \rangle
\quad - \langle x^*, N[J^*x_1 + x^*] - PN[J^*x_1 + x^*] \rangle
\quad = \|x_1\|^2 + \langle J^*x_1 + x^*, N(J^*x_1 + x^*) \rangle.
$$

At one point we have modified the domains of $J^*$ and $P$. But these operators are integrals so the modified operators are defined and $P$ is still such that a term vanishes.

Having a solution of (7), we know it is in $L^2[0, 1]$. So in taking it back
to (2), $P$ is again a projection. Then we check that the solution is in $C^2[0, 1]$ and hence a solution of (1).

By hypothesis (ii) and a variant of the Borsuk antipodal mapping theorem we conclude that there exists a solution of the equation $(I-R)z = 0$, i.e., there exists a solution of the boundary value problem (1).

**Remarks.** By applying the above method to the nonlinear problem

$$
(8) \quad x'' = f(t, x, x'), \quad x(0) = x(1), \quad x'(0) = x'(1)
$$

we conclude that (8) has at least one periodic solution if

(i) $f(t, x, y): [0, 1] \times R \times R \rightarrow R$ is continuous and $|f(t, x, y)| \leq \ a + b|x| + Q(|y|)$ where $Q$ is as in the theorem;

(ii) $(x, f(t, x, x')) + \|x\|^2 \geq 0$ for all $x$ satisfying $\|x\| = R$.

This result, proved by other methods, may be found in [1] or [4], for example.

We note that for problem (8), $T \neq T^*$. Also, the methods of the theorem may be applied to problems with other boundary conditions.

**Added in Proof.** Using that $(L_p^*, L_2, L_p)$ are in so-called normal position we can handle nonlinearities where $Q(s) = O(s^p)$, $p > 2$.

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, ST. LOUIS, MISSOURI 63121

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824