CONJUGATE SYSTEM CHARACTERIZATIONS OF $H^1$: COUNTER EXAMPLES FOR THE EUCLIDEAN PLANE AND LOCAL FIELDS

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ABSTRACT. The characterization of the Hardy space, $H^1$ of the plane, as those integrable functions whose first order Riesz transforms are (or whose maximal function is) integrable is well known. J.-A. Chao and M. Taibleson have shown that there is a conjugate system characterization of $H^1$ of a local field that parallels the Riesz system characterization of $H^1(R^2)$. C. Fefferman has conjectured that "nice" conjugate systems, such as the second order Riesz transforms would also give a characterization of $H^1(R^2)$. In the present paper a counter example of A. Gandulfo and M. Taibleson is described that shows that any conjugate system generated by an even kernel will fail to characterize $H^1$ of a local field. A counter example of J. Garcia-Cuerva is described that shows that the second order Riesz system for the Euclidean plane (which is generated by an even kernel) will fail to characterize $H^1(R^2)$ in the above sense.

Let $f \in L^1(R^n)$ and let $f^*(x) = \sup_{y>0} |f(x, y)|$, where $f(x, y)$ is the Poisson integral of $f$. We say that $f \in H^1(R^n)$ iff $f^* \in L^1(R^n)$. Let $(r, \theta)$ be the polar representation of $(x_1, x_2) \in R^2$, and let $(\cdot)^\wedge$ and $(\cdot)^\sim$ represents the Fourier transform and its inverse. The following characterization of $H^1(R^2)$ is in [5, §8]:

**Theorem A.** If $f$ is real-valued and $f \in L^1(R^2)$, then $f \in H^1(R^2)$ iff $(e^{i\theta} \hat{f})^\sim \in L^1(R^2)$.

Similarly, if $K$ is a local field, e.g., a $p$-adic field, we may define $f^*(x) = \sup_{k \in Z} |f(x, k)|$, where $f(x, k)$ is the regularization of $f$. (See [6, Chapter IV].) We say that $f \in H^1(K)$ iff $f^* \in L^1(K)$. The following characterization of $H^1(K)$ follows from results of Chao and Taibleson [3] and Chao [1], [2].

**Theorem B.** Suppose $\pi$ is a multiplicative character on $K$ that is unitary, ramified of degree 1, homogeneous of degree 0 and odd. If $f \in L^1(K)$ then $f \in H^1(K)$ iff $(\pi \hat{f})^\sim \in L^1(K)$.


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The "only if" part of the proof is in Chao [2]. The "if" part follows from [3, Theorem 2] and [2, Theorem 3.1 and example (i), p. 282].

The "if" part of the proof of Theorem B depends on the fact that \( n \) is an odd function. Taibleson and Gandulfo investigated this point and have shown Theorem B fails if \( n \) is even.

**Theorem 1.** Suppose \( \lambda \) is a multiplicative character on \( K \) that is unitary, ramified of degree 1, homogeneous of degree 0 and even. Then, there is a real-valued function \( g, g \in L^1(K) \) such that \( \lambda \hat{g} = \hat{g} \) and \( g^* \not\in L^1(K) \).

Thus, \( g \) and \( (\lambda \hat{g})^\sim \in L^1(K) \) but \( g \not\in H^1(K) \). If the local class field of \( K \) is odd and of order not equal to 3 (e.g., a \( p \)-adic field with \( p \neq 2 \) or 3) then there is a character \( \pi \) on \( K \) that satisfies the conditions of Theorem B while \( \pi^2 \) satisfies the conditions of Theorem 1. Note that \( f \rightarrow (\pi^2 \hat{f})^\sim \) is bounded from \( H^1 \) into itself (Chao [2]).

This result suggested that a similar investigation be made of the multiplier \( e^{2i\theta} \) on \( \mathbb{R}^2 \). Note that \( f \rightarrow (e^{2i\theta} \hat{f})^\sim \) is bounded from \( H^1 \) into itself (Fefferman and Stein [5, p. 190]). Recently it has been conjectured by Fefferman [4] that any "nice" multiplier should characterize \( H^1 \) in the sense of Theorem A. In particular, \( e^{2i\theta} \) is a usual example of such a "nice" multiplier. Garcia-Cuerva has investigated this problem and obtained the following result:

**Theorem 2.** There is a real-valued, radial function \( g, g \in L^1(\mathbb{R}^2) \) such that \( (e^{2i\theta} g)^\sim \in L^1(\mathbb{R}^2) \) but \( g \not\in H^1(\mathbb{R}^2) \).

We now briefly sketch proofs of Theorems 1 and 2.

**Lemma 1.** Let \( \lambda \) be as in Theorem 1. Then there exists a finite Borel measure \( \mu \), supported on \( \mathfrak{D} \) (the ring of integers in \( K \)) such that \( \mu \) is singular, \( \mu(\mathfrak{D}) = 0 \) and \( \lambda \hat{\mu} = \hat{\mu} \).

**Theorem 1** follows from Lemma 1. We note that \( \mu^* \in L^1 \), where \( \mu^*(x) = \sup \{ |\mu(x, k)| \} \). Also \( \sup |\mu(\cdot, k)|_1 < \infty \). Using the fact that \( \mu(x, k) \) is supported on \( \mathfrak{D} \times \mathbb{Z} \) we define \( f(x) = \sum_{k=-\infty}^{\infty} a_k \mu(x + c_k, k) \) where \( \{ c_k \} \) are coset representatives of \( \mathfrak{D} \) in \( K \). If \( \sum |a_k| < \infty \) we see that \( f \in L^1(K) \) and \( \lambda \hat{f} = \hat{f} \).

To construct the measure \( \mu \) we need to construct a regular function \( \mu(x, k) \) on \( K \times \mathbb{Z} \) such that \( \mu(x, k) \) is supported on \( \mathfrak{D} \times \mathbb{Z} \), \( \int_{\mathfrak{D}} \mu(x, k)dx = 0 \) for all \( k \), \( \| \mu(\cdot, k) \|_1 \leq A \) and \( \| \mu(\cdot, k) - \mu(\cdot, k - 1) \|_1 = B, k = -1, -2, \ldots \), for positive constants \( A \) and \( B \). (See [6, IV(1.8d) and (1.9b)].)

One now observes that if \( \chi \) is an additive character on \( K \) that is nontrivial on \( \mathfrak{D} \), but is trivial on \( \mathfrak{J} \) (the maximal ideal in \( \mathfrak{D} \)) then

\[
g(x) = \begin{cases} \text{Re} \chi(x), & x \in \mathfrak{D}, \\ 0, & x \notin \mathfrak{D}, \end{cases}
\]
has the property that \( \lambda \hat{g} = \hat{g} \) whenever \( \lambda \) is as in Theorem 1, \( \mu(x, k) \) is constructed by "patching together" various translations and dilations of \( g \).

For a sketch of the proof of Theorem 2 we will identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) in the usual way: \( (x_1, x_2) \leftrightarrow re^{i\theta} = z \).

For \( f \in L^1(\mathbb{C}) \) let \( \tilde{f}(\omega) = \text{P.V.} \int_C f(w - z) \frac{dz}{z^2} \). Then, \( \tilde{(f)} = e^{2i\theta} \tilde{f} \).

We now assume that \( f \) is radial; i.e., \( f(z) = g(|z|) \) for some \( g \). We then show that if \( f \) is radial on \( \mathbb{C} \) then \( f \in H^1(\mathbb{C}) \) iff \( rg(r) \in H^1 \) where \( rg(r) \) can be viewed as either a function defined on \([0, \infty)\) or as an even function on \( \mathbb{R} \). Finally we show that

\[
\tilde{f}(re^{i\theta}) = \pi e^{2i\theta} \left\{ \frac{2}{r^2} \int_0^r g(s)s \, ds - g(r) \right\}.
\]

Thus, we see that we need to find a function \( \varphi \in L^1(0, \infty) \) such that

\[
\frac{1}{r} \int_0^r \varphi(s) \, ds \in L^1(0, \infty) \quad \text{but} \quad \varphi \in H^1(0, \infty).
\]

Let \( I_{[a,b]} \) be the characteristic function of the interval \([a, b]\), and let \( l_k = k I_{[k,k+1/k]} - (1/k)I_{[k+1/k,k+2/k]} \). We see that \( \int_0^\infty |l_k| = 2, \int_0^1 l_k = 0, \int_0^\infty (1/r) \int_0^r l_k |dr| < 1 \). We see that there is a \( C > 0 \) such that if \( k \) is large enough \( n_0 = \sqrt[k]{\ln k} \geq C \ln k \). A little calculation shows that if \( n_0 \) is large enough, then \( \varphi = \sum_{n=0}^{\infty} (1/n^2)I_{n_0 \leq n} \) has the required properties.

As a final comment, we observe that the formula for \( \tilde{f}, f \) integrable and radial extends easily to finite Borel measures that are radial. Apply that result to the singular measure \( \mu \) that has measure 1 uniformly distributed on the unit circle in \( \mathbb{C} \) and measure \(-1\) uniformly distributed on the circle of radius two. It is easy to check that \( \tilde{\mu} \) is a singular measure. Together with the result of Lemma 1 we see that the conjugate systems induced by the multipliers \( e^{2i\theta} \) and \( \pi^2 \) (on the Euclidean plane or local fields respectively) fail to produce an F. and M. Riesz theorem in the sense: There is a finite Borel measure \( \mu \), such that the conjugate of \( \mu \) is also a finite measure, but \( \mu \) is not absolutely continuous.

REFERENCES


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