VERSAL UNFOLDINGS OF $G$-INARIANT FUNCTIONS

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1. We announce here some results on equivariant local differential analysis. The proofs will appear elsewhere [7]. We consider a compact Lie group $G$, acting orthogonally on $R^n$. $C^\infty(x)$ (respectively $C^\infty(R^n)$) will denote the ring of germs of $C^\infty$ functions around $0 \in R^n$ (the ring of $C^\infty$ functions of $R^n$). The germ of $R^n$ at $0$ will be denoted by $X$. $C^\infty(x)^G$, $C^\infty(R^n)^G$ will denote the $G$-invariant germs (functions). We shall consider parameter (germs of) spaces $U$, $V$, . . . , on which $G$ acts, by definition, trivially.

If $f(x) \in C^\infty(x)^G$, an unfolding of $f(x)$ is an $F(x, u) \in C^\infty(x, u)^G$ such that $F(x, 0) \equiv f(x)$. The unfolding $F(x, u)$ is versal, if any other unfolding of $f(x)$, $H(x, v) \in C^\infty(x, v)^G$, can be induced from $F$, by a commutative diagram

\[
\begin{array}{ccc}
X \times V & \xrightarrow{\Phi} & X \times U \\
\downarrow & & \downarrow \\
V & \xrightarrow{\varphi} & U
\end{array}
\]

such that:

(a) $\Phi, \varphi \in C^\infty$,
(b) $\Phi$ is $G$-equivariant,
(c) $\Phi|X \times 0 \equiv \text{id} X$,
(d) $H = F \circ \Phi$.

$G$ also acts on smooth vector-fields on $X(R^n)$. We consider the invariant (germs of) vector-fields $\Gamma^\infty(TX)^G \subset \Gamma^\infty(TX)$ i.e., fields such that $g\xi(x) = Tg(\xi(x)) = \xi(gx)$. $\Gamma^\infty(TX)^G$ is a $C^\infty(x)^G$-module moreover, if $f(x) \in C^\infty(x)^G$, the subset

$$J_G(f) = \{df(\xi), \xi \in \Gamma^\infty(TX)^G\} \subset C^\infty(x)^G.$$ 

is an ideal, called the $G$-jacobian ideal of $f$. We shall assume that $f$ is given, and that $\dim_R C^\infty(x)^G/J_G(f) < \infty$.

By definition $F(x, u) \in C^\infty(x, u)^G$, unfolding of $f$, is infinitesimally versal if the images of $\partial F(x, 0)/\partial u_1, \ldots, \partial F(x, 0)/\partial u_k$ in $C^\infty(x)^G/J_G(f)$ generate the $R$-vector space $C^\infty(x)^G/J_G(f)$.

Theorem 1. If the unfolding $F(x, u) \in C^\infty(x, u)^G$ (of $f(x) \in C^\infty(x)^G$) is infinitesimally versal, it is versal. □

This is a generalization of a result of J. Mather [5], R. Thom [16], V. M. Zakalyukin [14], F. Sergeraert [10], G. Lassalle [3], and others.

This theorem should be useful for “catastrophe theory in the presence of symmetry” [11], [12].

2. The main ingredient for proving Theorem 1 is the equivariant preparation theorem, which we describe now.

Suppose $G$ (compact Lie group) acts orthogonally on $\mathbb{R}^n, \mathbb{R}^p$; the germs of these two spaces, around $0$, will be denoted by $X, Y$.

We consider a germ of smooth map $f \in C^\infty(X, Y)$ which is equivariant: $f(gx) = gf(x)$. Then $f$ induces a local ring homomorphism $C^\infty(x)^G \leftarrow C^\infty(y)^G$.

Theorem 2. If $M$ is a finitely generated $C^\infty(x)^G$-module, such that

$$\dim_\mathbb{R} M/f^*MC^\infty(y)^G < \infty,$$

then $M$ is also finitely generated as a $C^\infty(y)^G$-module. □

This is a generalization of a theorem of B. Malgrange [4] and J. Mather [6].

3. This paragraph provides some examples for Theorem 1.

With $G$ compact as before we consider the algebra of $G$-invariant polynomials $\mathbb{R}[x]^G$. By a classical result of Hilbert [2], [13], this algebra is finitely generated, i.e. there is a polynomial map $y = \rho(x) (\mathbb{R}^n \overset{\rho}{\rightarrow} \mathbb{R}^p)$ (given by finitely many homogenous polynomials, of positive degree), such that $\mathbb{R}[x]^G \leftarrow R^* y$ is surjective. It had been conjectured, for some time, that this is still true in the $C^\infty$ case. In fact G. Glaeser [15] had proved it for $G =$ the symmetric group, and for some time at least the local case for finite $G$ has been known to result from the preparation theorem (see for example [1]).

Note also that there is a way to work along the diagonals and go from the local to the global case. Now, the general compact case has been proved by G. Schwarz [9], and it is this result which makes the present paper possible.

We hope to be able to complete the details of a different proof, in some future (including, possibly, the $C^k$-case). Since Hilbert’s XIVth problem is solved negatively, the noncompact case is hopeless.

Now if $\xi$ is a smooth $G$-invariant vector field on $\mathbb{R}^n$, one has in a natural way, a direct image of $\xi$: $\rho_* \xi$, which is a continuous vector field on the semialgebraic subset $\rho R^n \subset \mathbb{R}^p$.

Proposition 3. If $\xi \in \Gamma^\infty(TR^n)^G$, then there is a smooth ($C^\infty$) vector field $\eta \in \Gamma^\infty(TR^p)$ such that $\eta|_\rho R^p \equiv \rho_* \xi$. □

The same result is true for germs, and we deduce that if $\varphi(y) \in C^\infty(y)$,
and \( J(\varphi) \subset C^\infty(y) \) is the usual jacobian ideal of \( \varphi \), then \( \rho^*J(\varphi) \supset J_G(\rho^*\varphi) \).
(Note that \( \rho^*\varphi \in C^\infty(x)^G \).) This leads to one way of finding elements of finite codimension in \( C^\infty(x)^G \). A better way is given by the following

**Proposition 4.** Let \( f(x) \in C^\infty(x)^G \subset C^\infty(x) \) such that
\[
\dim_R C^\infty(x)/J(f) < \infty.
\]

Let \( \varphi_1(x), \ldots, \varphi_k(x) \in C^\infty(x) \) be generators of \( C^\infty(x)/J(f) \), as a vector space. Then \( C^\infty(x)^G/J_G(f) \) is a finite dimensional vector space, generated by the averages of the \( \varphi_i \)'s:
\[
\psi_i(x) = \int_G \varphi_i(gx) \, d\mu(g) \in C^\infty(x)^G.
\]

Here \( d\mu(g) \) is the Haar measure of \( G \). The general idea behind all this is that once one has a smooth version of Hilbert's finiteness theorem from the classical invariant theory, the Thom-Mather type theory of singularities can be extended to the case when a compact Lie group is operating. We plan to develop stability theory on these lines (see also \[8\]).

**BIBLIOGRAPHY**