DIRECT SUM PROPERTIES
OF QUASI-INJECTIVE MODULES

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Communicated by Barbara Osofsky, October 15, 1975

Abstract. A functorial method is described by which certain problems can be transferred from quasi-injective modules to nonsingular injective modules. Applications include the uniqueness of \( n \)th roots: If \( A \) and \( B \) are quasi-injective modules such that \( A^n \cong B^n \), then \( A \cong B \).

All rings in this paper are associative with unit, all modules are unital right modules, and endomorphism rings act on the left. The letter \( R \) denotes a ring. We use \( \mathcal{J}(\cdot) \) to denote the Jacobson radical.

Recall that a module \( A \) is quasi-injective provided any homomorphism of a submodule of \( A \) into \( A \) extends to an endomorphism of \( A \). For example, all injective modules and all semisimple (completely reducible) modules are quasi-injective.

**Theorem 1.** Let \( A \) be a quasi-injective right \( R \)-module, and set \( Q = \text{End}_R(A) \). Then \( Q/\mathcal{J}(Q) \) is a regular, right self-injective ring, and idempotents can be lifted modulo \( \mathcal{J}(Q) \).

**Proof.** Regularity and idempotent-lifting were proved by Faith and Utumi [2, Theorems 3.1, 4.1]. Self-injectivity was proved by Osofsky [6, Theorem 12] and Renault [7, Corollaire 3.5].

**Proposition 2.** Let \( A \) be a quasi-injective right \( R \)-module, and set \( Q = \text{End}_R(A) \). Let \( \mathcal{U} \) denote the category of all direct summands of finite direct sums of copies of \( A \), and let \( \mathcal{P} \) denote the category of all finitely generated projective right \( (Q/\mathcal{J}(Q)) \)-modules. Then there exists an additive (covariant) functor \( F: \mathcal{U} \rightarrow \mathcal{P} \) with the following properties.

(a) For all \( B, C \in \mathcal{U} \), the induced map \( \text{Hom}_\mathcal{U}(B, C) \rightarrow \text{Hom}_\mathcal{P}(F(B), F(C)) \) is surjective.

(b) Given any \( P \in \mathcal{P} \), there exists \( B \in \mathcal{U} \) such that \( F(B) \cong P \).

(c) A map \( f \in \mathcal{U} \) is an isomorphism if and only if \( F(f) \) is an isomorphism in \( \mathcal{P} \).

**Proof.** If \( \mathcal{P}_0 \) denotes the category of all finitely generated projective right \( Q \)-modules, then \( \text{Hom}_\mathcal{R}(A, \cdot) \) defines a category equivalence \( G: \mathcal{U} \rightarrow \mathcal{P}_0 \). Second, \( (\cdot) \otimes_Q (Q/\mathcal{J}(Q)) \) gives us an additive functor \( H: \mathcal{P}_0 \rightarrow \mathcal{P} \), and we set \( F = HG \).
Properties (a) and (c) hold without any hypotheses on $A$, while (b) follows from the regularity of $Q/J(Q)$ and the fact that idempotents lift modulo $J(Q)$. □

Over a regular, right self-injective ring, all finitely generated projective right modules are injective and nonsingular. Thus the functor $F$ in Proposition 2 enables us to transfer problems from the quasi-injective module $A$ to the nonsingular injective module $F(A)$.

**Theorem 3.** Let $A, B$ be quasi-injective right $R$-modules, and let $n$ be a positive integer.

(a) If $A^n$ is isomorphic to a direct summand of $B^n$, then $A$ is isomorphic to a direct summand of $B$.

(b) If $A^n \cong B^n$, then $A \cong B$.

**Proof.** Setting $Q = \text{End}_R(B)$, we use Proposition 2 to transfer the problem to nonsingular injective right $(Q/J(Q))$-modules, where the required properties follow from [5, Proposition 9.1].

**Definition.** A module $A$ is **directly finite** provided $A$ is not isomorphic to any proper direct summand of itself.

**Theorem 4** [1, Proposition 5]. Let $A$ be a directly finite quasi-injective right $R$-module. If $B$ and $C$ are any right $R$-modules such that $A \otimes B = A \otimes C$, then $B \cong C$.

**Proof.** If $P$ is any directly finite nonsingular injective module, then [8, Corollary 8] (or [5, Theorem 3.8]) shows that isomorphic direct summands of $P$ have isomorphic complements. Using Proposition 2, the module $A$ has the same property. In addition, [3, Theorem 3] shows that $A$ has the exchange property, hence cancellation follows from [4, Theorem 2].

**Corollary 5.** If $A_1, \ldots, A_n$ are directly finite quasi-injective right $R$-modules, then $A_1 \oplus \cdots \oplus A_n$ is directly finite (but not necessarily quasi-injective).

**Proof.** Obviously cancellation carries over from the $A_i$ to their direct sum. On the other hand, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Q}$ are directly finite quasi-injective $\mathbb{Z}$-modules whose direct sum is not quasi-injective. □

**Theorem 6.** If $A$ is a quasi-injective right $R$-module, then there exists a decomposition $A = B \oplus C$ such that $B$ is directly finite and $C \cong C^2$.

**Proof.** The corresponding decomposition for nonsingular injective modules is given by [5, Proposition 8.4 and Theorem 7.2]. □

**Corollary 7.** Let $A$ be a quasi-injective right $R$-module. Then $A$ is directly finite if and only if $A$ has no nonzero direct summands $C$ for which $C \cong C^2$. □
REFERENCES


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