In this note we define "homology groups" relative to the flat site, and list some of their properties, in the case that the base scheme is algebraic over a field.

$X_{fl}$ denotes the big f.p.p.f. site over a scheme $X$ and $S(X_{fl})$ the corresponding category of sheaves. $S = \text{spec } k$, where $k$ is a field of characteristic $p$. $A(al)$ denotes the category of commutative algebraic group schemes over $S$ and $A(u, f)$ $\supset A(u) \supset A(uf) \supset A(f)$ the subcategories consisting of those affine groups which are respectively unipotent or finite, unipotent, unipotent and finite, finite. The letter $A$ always stands for one of these categories and Pro-$A$ for the corresponding pro-category. The notations for derived categories are as in [6].

1. **Theorem** (Universal Coefficient Theorem). For any morphism $\pi: X \to S$ of finite type and any $A$, there exists a complex $L_s(X/S, A)$ in $K^-(\text{Pro}-A)$ such that:
   (a) $L_s(X/S, A)$ is a projective object, all $s$;
   (b) $\text{Hom}_{\text{Pro}-A}(L_s(X/S, A), N) \cong R\pi_*N_X$ in $D^+(S(S_{fl}))$ for all $N$ in $A$.

Moreover, $L_s(X/S, A)$ is unique, up to isomorphism, in $K^-(\text{Pro}-A)$.

**Proof.** Choose a conservative family of points for $X_{fl}$, and let $C^*(F)$ be the corresponding Godement resolution of a sheaf $F$ [1, XVII 4.2]. Choose $L_s$ to pro-represent the functor $N \mapsto \Gamma(X, C^s(N_X)): A \to Ab$.

2. **Corollary.** Write $H_s(X/S, A)$ for $H_s(L_s(X/S, A))$. There is a spectral sequence

   \[ \text{Ext}_{\text{Pro}-A}^r(H_s(X/S, A), N) \Rightarrow H^{r+s}(X_{fl}, N_X) \]
   for all $N$ in $A$.

3. **Definition.** $L_s(X/S, A)$ is the flat homology complex of $X/S$ relative to $A$, and $H_s(X/S, A)$ is the $s$th flat homology group.

4. **Remarks.** (a) Theorem 1 is basically as conjectured by Grothendieck [5, p. 316].
   (b) $L_s(X/S, A)$ and $H_s(X/S, A)$ are covariant functors in $X/S$.
   (c) If $\omega_0: A(al) \to A(f)$ is the functor taking a group scheme to its maximal finite quotient, then $\omega_0(L_s(X/S, A(al))) = L_s(X/S, A(f))$. Thus there

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1 Supported by NSF at Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France.
is a third-quadrant spectral sequence \( \omega_r(H^s_s(X/S, A(\mathcal{f}))) \Rightarrow H_{r+s}(X/S, A(\mathcal{f})) \) where \( \omega_r = L^r \omega_0. \)

5. **THEOREM.** Assume \( k \) to be algebraically closed and let \( M \) be the functor taking a group scheme to its Dieudonné module (in the sense of [4, III]). Then

\[
M(L_s(X/S, A(u, f))) = H_s(X_{\text{Zar}}, W_{f}) \oplus (H^s(X_{f1}, \mu_p^\infty) \otimes \mathbb{Z} W(k)),
\]

where \( W_n \) is the group scheme of Witt vectors of length \( n \) and \( W = \lim_{\to} W_n(O_X) \) and \( W(k) = \lim_{\to} W_n(k) \).

**PROOF.** Immediate from the definitions of \( L \) and \( M \).

6. **COROLLARY.** \( M(H^s_s(X/S, A(u, f))) = H^s_s(X_{\text{Zar}}, W_{f}) \oplus H^s_s(X_{f1}, \mu_p^\infty) \otimes \mathbb{Z} W(k) \).

**PROOF.** "\( \lim_{\to} \)" \( W_n \) and "\( \lim_{\to} \)" \( \mu_p^n \) behave as injectives in \( \mathbb{A} \).

7. **REMARK.** \( M(H_{-}) \) is equal to the group \( I(X) \) studied in [7, §4].

8. **THEOREM.** Assume \( k \) to be algebraically closed and \( X/S \) to be proper. Then \( L_s(X/S, A(u)) \) is isomorphic (in \( K^- (\text{Pro-} A(u)) \)) to \( L_s(X/S, A(uf)) \).

**PROOF.** \( H^s_s(X/S, A(u)) \in \text{Pro-} A(uf) \) for otherwise \( H^s_s(X, O_X) \) would have infinite dimension over \( k \).

9. **THEOREM.** Write \( N^- \) for the formal group associated to an affine group scheme \( N \) by Cartier duality (see [4, II.4]), and write \( H^s_s \) for \( H^s_s(L_s) = H^s(L_s^-) \) where \( L_s^- = L_s(X/S, A(u)) \). Then \( H^s_s \) is a connected formal group of finite-type (see [4, p. 35]) and represents the functor of finite \( S \)-schemes.

\[
T \mapsto \ker(T, R^s_{\pi*} G_m) \rightarrow \Gamma(T_{\text{red}}, R^s_{\pi*} G_m))
\]

**PROOF.** Regard \( U = \ker(G_m, T \mapsto G_{m, T_{\text{red}}}) \) as a sheaf on \( T_{\text{red}} \), and use (8).

10. **COROLLARY.** Write \( \Phi^s(T) = \ker(H^s_s(X_T, G_m) \rightarrow H^s_s(X_{T_{\text{red}}}, G_m)) \). If \( \Phi^s^{-1} \) is a formally smooth functor then \( \Phi^s \) is represented by a formal group.

**PROOF.** Immediate from the theorem.

11. **REMARKS.** (a) Intuitively (9) says that \( L^- \) represents \( R^- \pi* G_m \) infinitesimally.

(b) Generalizations of (10), but not (9), may be found in [2].

12. **THEOREM.** Assume that \( k \) is algebraically closed, \( X \) is projective and smooth over \( k \), and \( p > \dim(X) \). Then

\[
\text{Hom}_W(K/W, M(H^s_s(X/S, A(f)))) \otimes_W K \cong (H^s_s(X/W, O_{X/W}) \otimes_W K)_{[0,1]}
\]
as \( F \)-isocrystals, where \( W = W(k) \), \( K = \) field of fractions of \( W \), and the right-hand term is the part of crystalline cohomology with slopes between 0 and 1 (inclusive).

**Proof.** Follows from [3] and (6).

13. **Remarks.** (a) The last theorem states that (modulo torsion) the knowledge of the flat cohomology of finite constant group schemes on \( X \) is equivalent to the knowledge of the part of crystalline cohomology with slopes between 0 and 1.

(b) (12) differs from the “hope” expressed by Grothendieck [5, p. 316].

**BIBLIOGRAPHY**


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