ON $L^1$ CONVERGENCE OF CERTAIN COSINE SUMS

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Abstract. It is shown that to a certain cosine series $f$, a Rees-Stanojević cosine sum $g_n$ can be associated such that $g_n$ converges to $f$ pointwise, and a necessary and sufficient condition for $L^1$ convergence of $g_n$ to $f$ is given. As a corollary to that result we have a generalization of the classical result of this kind. Other corollaries are given concerning the well-known integrability conditions.

This paper gives an analogue for modified cosine sums of the classical result concerning $L^1$ convergence of a Fourier sine series. Rees and Stanojević [1] introduced these cosine sums that approximate their pointwise limit “better” than the classical cosine series since they converge in the $L^1$ metric space to their limit when the classical cosine series may not.

**LEMMA 1.** Let $f(x) = \lim_{n \to \infty} f_n(x)$ where $f_n(x) = \frac{1}{2}a(0) + \sum_{k=1}^{n} a(k)\cos kx$, \[ \lim_{n \to \infty} a(n) = 0, \quad \text{and} \quad \sum_{k=0}^{n} |\Delta a(k)| < \infty. \] Let $g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \Delta a(k) + \sum_{k=1}^{n} \sum_{j=k}^{\infty} \Delta a(j)\cos kx$. Then $\lim_{n \to \infty} g_n(x) = f(x)$.

**THEOREM 1.** Let $f$, $f_n$, and $g_n$ be as defined in Lemma 1. Then $g_n$ converges to $f$ in the $L^1$ metric if and only if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_{0}^{\infty} |\Delta a(k)D_k(x)| < \epsilon$ for all $n \geq 0$, where $D_k(x)$ is the Dirichlet kernel.

**COROLLARY 1.** Let $f_n$ and $f$ be as defined in Lemma 1. If for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\int_{0}^{\infty} |\sum_{k=n+1}^{\infty} \Delta a(k)D_k(x)| < \epsilon$ for all $n \geq 0$ then $f_n$ converges to $f$ in the $L^1$ metric if and only if $\lim_{n \to \infty} a(n)\log n = 0$.

**COROLLARY 2.** Let $f$ and $g_n$ be as defined in Lemma 1. If $\sum_{n=1}^{\infty} |\Delta^2 a(n)|(n+1) < \infty$, then $g_n$ converges to $f$ in the $L^1$ metric.

**COROLLARY 3.** Let $f$ and $g_n$ be as defined in Lemma 1. If $\sum_{k=1}^{\infty} |\Delta a(k)|\log k < \infty$, then $g_n$ converges to $f$ in the $L^1$ metric.

**COROLLARY 4.** Let $f$ and $g_n$ be as defined in Lemma 1. If $a(n) = b(n) + c(n)$ where $\lim_{n \to \infty} b(n) = \lim_{n \to \infty} c(n) = 0$, $\sum_{n=1}^{\infty} |\Delta b(n)|\log n < \infty$, and $\sum_{n=1}^{\infty} |\Delta^2 c(n)|(n+1) < \infty$, then $g_n$ converges to $f$ in the $L^1$ metric.

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Corollary 5. Let $f$ and $g_n$ be as defined in Lemma 1. If $a(n) = \alpha(n)\beta(n)$ where $\sum_{n=1}^{\infty} |\Delta \alpha(n)| < \infty$, $|\beta(n)| \leq M$, $\sum_{n=1}^{\infty} |\Delta^2 \beta(n)|(n + 1) < \infty$, and $\sum_{n=1}^{\infty} |\beta(n)\Delta \alpha(n)| \log n < \infty$, then $g_n$ converges to $f$ in the $L^1$ metric.

Proofs and details of these results will appear elsewhere.

Reference