THE TOTAL CURVATURE OF KNOTTED SPHERES

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Chern and Lashof [1] proved several inequalities concerning the total curvature of an immersed manifold. Their second result is a weak generalization of the Fary-Milnor theorem [2], [5] for closed space curves. In this paper, a stronger result (Corollary 1), the complete homotopy extension, is stated and proved. I would like to thank Bill Pohl for conversations surrounding the formulation and proof of this result.

I. Background. Let \( x: M^n \to E^{n+N} \) be a \( C^\infty \)-immersion into Euclidean space of dimension \( n+N \) \((N > 0)\); and \( B_v \) be the bundle of unit normal vectors of \( x(M^n) \). A point of \( B_v \) is a pair \((p, \nu(p))\), where \( \nu(p) \) is a unit normal vector to \( x(M^n) \) at \( x(p) \). The map \( \bar{\nu}: B_v \to S^{n+N-1}_0 \), into the unit sphere of \( E^{n+N} \), is defined by \( \bar{\nu}(p, \nu(p)) = \nu(p) \).

The Lipschitz-Killing curvature [1], \( G(p, \nu) \) at \( \nu(p) \), is then given by the \( \bar{\nu} \)-ratio of corresponding volume elements in \( S^{n+N-1}_0 \) and \( B_v \). The total curvature of \( M^n \) at \( p \) is \( K^*(p) = \int |G(p, \nu)| \, d\sigma \), the integral being taken over the sphere of unit normal vectors at \( x(p) \). The total curvature of \( M^n \) is given by \( K^* = K^*(M) = \int_{\partial M} K^*(p) \, dV \).

The first two Chern-Lashof theorems can be stated as: Given \( M^n \) compact without boundary, and \( c(m) \) the area of the unit hypersphere \( S^m_0 \subset E^{m+1} \), then:

**Corollary 1.** \( K^*(M) \geq 2c(n+N-1) \).

**Corollary 2.** If \( K^*(M) < 3c(n+N-1) \), then \( M \) is homeomorphic to \( S^n \).

The essential argument of their proof can be summarized as a lemma.

**Lemma 1.** If, for almost all \( \nu_0 \in S^{n+N-1}_0 \), the height function \( \langle \nu_0, - \rangle: x(M) \to \mathbb{R} \) has at least \( k \) distinct critical points, then \( K^*(M) \geq kc(n+N-1) \).

Their method is an adaptation of the technique used by Fenchel [3]. This fact suggested that Corollary 2 is a weak generalization of Fary-Milnor.

II. The main result. In this section, a curvature inequality is given which distinguishes between different knottings of \( S^n \). The method, based on Chern-Lashof, takes off from a remark of Fox [4] in which P. L. approximations yield the corresponding \( S^1 \) result.

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For simplicity of presentation, attention is restricted to knotted spheres; that is, \( M^n = S^n \) and codimension \( N = 2 \). Recall, for a mapping \( x: S^n \to E^{n+2} \), the group of the map is \( \pi(x) = \pi_1 [E^{n+2} - x(S^n)] \).

**Definition 1.** \( g(x) \) = the minimal number of generators needed to present \( \pi(x) \).

**Theorem I.** \( K^*(S^n) \geq 2g(x)c(n + 1) \).

**Corollary 1.** If \( K^*(S^n) < 4c(n + 1) \), then \( \pi(x) = \mathbb{Z} \).

The corollary follows trivially since any \( \pi(x) \) has \( \mathbb{Z} \) as a subgroup. Theorem I is a consequence of Lemma 1 combined with the obvious.

**Proposition 1.** For almost all \( v_0 \in S^{n+1}_0 \), the height function \( \langle v_0, - \rangle : x(S^n) \to \mathbb{R} \) has at least \( 2g(x) \) distinct critical points.

**Proof.** Since we only need to account for an open dense subset of the \( v_0 \)'s, fix a height \( \langle v_0, - \rangle \) which is Morse. Choose a basepoint, \( * \), which is "higher" than \( x(S^n) \). The proposition is shown by constructing a canonical set of generators for \( \pi(x, *) \), and deforming an arbitrary loop, \( \gamma \in \pi(x, *) \), into a sum of these. The deformation is first described. The required generating set will be obvious at the outcome.

Since \( * \) is higher than \( x(S^n) \), assume that the loop \( \gamma \) is strictly lower than \( * \). Now, define a lifting-homotopy as a homotopy \( H(x, t) \) which is always moving to higher levels, that is, one where \( \langle v_0, H(x, t) \rangle \) is nondecreasing in \( t \) for all fixed \( x \) in the loop parametrization. The problem involved is to determine the obstructions in \( x(S^n) \) preventing \( \gamma \) from being pulled up all the way. Clearly, any such phenomenon will be local. The crucial observation is that \( \gamma \) can only be "caught" on maximums of \( \langle v_0, - \rangle : x(S^n) \to \mathbb{R} \).

Take a collection of open collared balls, \( U_i \subset W_i \), in \( E^{n+2} \) such that: (1) \( \{U_i\} \) is a finite covering of a simply-connected volume enclosing \( x(S^n) \); (2) each critical point \( p \) is contained in only one ball \( W_j \); and (3) there are Morse-coordinates for \( (W_i \cap x(S^n)) \) whose axes are strictly monotonic w.r.t \( \langle v_0, - \rangle \). Clearly, any part of \( \gamma \) lying in a \( U_i \) not containing a critical point can be lifted out of the ball. This means that attention can be focused on the \( U_1, \ldots, U_k \) containing \( p_1, \ldots, p_k \).

Now, suppose that \( p_j \) is not a maximum. Then the height function is increasing on at least one Morse-axis, and the piece of \( x(S^n) \) locally obstructing \( \gamma \) has at least codim 3. There are index(\( p_j \)) > 0 degrees of freedom with which to translate a segment of \( \gamma \) and lift it into the collar \( (W_j - U_j) \) such that it lies above \( U_j \cap x(S^n) \). After a finite number of such movements, \( \gamma \) will only be obstructed by balls containing maximums.

Next, assign a unique 'canonical' element of \( \pi(x) \) to each maximum. For \( p_j \) a maximum, fix a loop \( \gamma_j \) which passes under \( p_j \) only once. This can be ar-
ranged (inside $\mathcal{W}_j$) by adding a lower hemisphere to $U_j \cap x(S^n)$, and taking $\gamma_j$ as a generator which leaves $\mathcal{W}_j$ through the north pole and is increasing till *. Any segments of $\gamma$ stuck in $U_j$ can be lined up (inside $\mathcal{W}_j$) with $\gamma_j$. The rest of the loop goes up and away. Hence, the collection $\{\gamma_j\}$ is a set of generators for $\pi(x)$.

Summarizing, any $\langle v_0, - \rangle$ has at least $g(x)$ maximums. Next, if $C_i$ = the number of critical points of index $i$, then the Morse equality gives: (1) for $n$ odd, $\sum_{i=1}^{n} (-1)^i C_i = C_0 \geq g(x)$, and there are at least $g(x)$ critical points other than maximums; (2) for $n$ even, there is at least one minimum, hence:

$\sum_{i=1}^{n} (-1)^i C_i = C_0 + C_n - 2$, and there are at least $(g(x) - 1)$ critical points other than maximums and minimums. In either case, the proof is complete.

BIBLIOGRAPHY


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