CONFORMAL MAPS ON HILBERT SPACE

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1. Introduction. In [1] Nevanlinna gave a simple proof of the following theorem of Liouville. (Precise definitions appear below.)

THEOREM 1. Suppose \( U \) is a connected open set in a real Hilbert space \( H \) of dimension \( \geq 3 \) (including \( \infty \)) and \( f: U \to H \) is \( C^4 \) and conformal. Then \( f \) is either

(a) an affine map whose linear part is a constant multiple of a unitary operator,

(b) an inversion with respect to a sphere,

(c) \( f_1 \circ f_2 \) where \( f_1 \) is of type (a) and \( f_2 \) is of type (b).

REMARKS. (i) The dimension of \( H \) must be \( \geq 3 \) because every holomorphic map on \( \mathbb{C} \) with a nowhere zero derivative is conformal.

(ii) For \( \mathbb{R}^n \), the theorem is known even for \( f \) just \( C^1 \) [2].

(iii) The proof of Nevanlinna depends on \( f \) being \( C^4 \).

In this paper we outline how a technique in [3], when recognized as applying to conformal mappings and suitably modified, can be used to prove the theorem with \( f \) only \( C^3 \).

2. Notation and definitions. \( H \) will be a real infinite dimensional Hilbert space and \( U \) a connected open subset. A map is \( C^n \) if it is \( n \) times continuously Fréchet differentiable as in [4]. A \( C^1 \) function \( f: U \to H \) is called conformal if \( Df_x \) is a linear isomorphism and there is a function \( c: U \to \mathbb{R} \) such that

\[
\langle Df_x(h_1), Df_x(h_2) \rangle = c(x) \langle h_1, h_2 \rangle
\]

for all \( x \) in \( U \) and all \( h_1, h_2 \) in \( H \). (This definition is merely a reformulation of the more geometric definition that says \( f \) preserves the angle between two curves meeting at a point.) Banach and Hilbert manifolds are defined as in [4].

By an inversion with respect to the sphere \( \{ x \in H: \| x - p \| = r \} \) I mean the map \( x \to x' \) where

(i) \( \| x - p \| \| x' - p \| = r^2 \) and

(ii) \( x \) and \( x' \) lie on the same ray originating at \( p \). The analytic form of such an inversion is

\[
x \to r^2(x - p)\| x - p \|^{-2} + p.
\]
3. **Outline of the proof.** (1) We develop the theory of connections for Banach manifolds and specialize to the case of Riemannian connections for a $C^3$ Hilbert manifold. For each chart $(W, \psi: W \to H)$ of the manifold a $C^1$ function $\Gamma$, called the Christoffel function, is defined on $\psi(W)$ such that $\Gamma(y)$ is a continuous $H$-valued bilinear map on $H$ for each $y$ in $\psi(W)$. The collection of such $\Gamma$ (together with a coherence property on the overlap of charts) determines and is determined by the connection.

(2) Let $d(x) = 1/\sqrt{c(x)}$. (Since $Df_x$ is one-one, $c(x)$ is not zero.) For example if $f$ is the affine map $f(x) = rL_0(x) + h_0$ where $r$ is real, $L_0$ unitary and $h_0 \in H$, we have $d(x) = 1/r$. On the other hand for the inversion

$$f(x) = r^2(x-p)\|x-p\|^2 + p$$

we have $d(x) = \langle x - p, x - p \rangle/r^2$.

Since Hilbert space with the inner product as Riemannian metric has zero curvature we get the following equation for $d$:

(*) \[ 2D^2d_x(h_1, h_2) = 2Dd_x(h_1)Dd_x(h_2) + Dd_x[\Gamma_x(h_1, h_2)]. \]

To derive this we use the fact that the dimension of $H$ is $\geq 3$.

(3) We prove that in a neighborhood of each point $x_0$, the above equation has a unique solution

(\**\) \[ d(x) = A (x - x_0, x - x_0) + \langle b, x - x_0 \rangle + C \]

where $C = d(x_0)$, $b$ is the element in $H$ corresponding to $Dd_{x_0}$ under the canonical isomorphism of $H$ with its dual $H^*$ and $A = \langle b, b \rangle/4C$.

The method of proof is to start at $x_0$ where (\**) is true and then to show that equality continues as we move in any direction. Pick a unit vector $u$ and define $g_1(t) = d(x_0 + tu)$. Using (*) the function $K_1(t) = [g_1(t), Dg_1(t)] \in R \times H^*$ is shown to satisfy a differential equation of the form $K'(t) = F[t, K(t)]$ with initial condition $K(0) = [d(x_0), Dd_{x_0}]$. Letting $g_2(t) = A (tu, tu) + \langle b, tu \rangle + C$ we verify that $K_2(t) = [g_2(t), Dg_2(t)]$ satisfies the same differential equation and initial condition. The equality of $g_1$ and $g_2$ follows from uniqueness.

(4) Using the connectedness of $U$ we get that the local solution in (3) is actually a global solution for $d$.

(5) We show that if $f: U \to H$ and $g: U \to H$ are $C^3$ conformal maps such that $g$ is one-one and $d_f = d_g$ (where $d_f$ is the $d$ corresponding to $f$), then there is a vector $h$ in $H$ and unitary operator $L$ such that $f = L \circ g + h$.

(6) From (4) we know that $d(x) = A (x - x_0, x - x_0) + \langle b, x - x_0 \rangle + C$.

**Case 1.** $b = 0$ and $A = \langle b, b \rangle/4C = 0$ in which case $d(x) = C$ has the same form as the $d$ for an affine map as in (2), if $C = 1/r$.

**Case 2.** $b \neq 0$ and thus $A \neq 0$. Then $d(x) = \langle x - p_0, x - p_0 \rangle/r$ where $r = 4C/|b, b|$ and $p = x_0 + 2Cb/(b, b)$. This is the same form as the $d$ for an inversion as in (2) above.
(7) Combining (6) with (5) we get our theorem.

REFERENCES


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