FINITE HJELMSLEV PLANES
WITH NEW INTEGER INVARIANTS

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Communicated March 28, 1975

Projective Hjelmslev planes (PH-planes) are a generalization of projective planes in which each point-pair is joined by at least one line and, dually, each line-pair has a nontrivial intersection. Multiply joined points (and multiply intersecting lines) are called neighbor points (and neighbor lines). By hypothesis, the neighbor relations of a PH-plane \( A \) are equivalence relations which induce a canonical epimorphism from \( A \) to a projective plane \( \overline{A} \). If \( A \) is finite, there exists [4] an integer \( t \) such that the inverse image of every point and every line of \( \overline{A} \) contains precisely \( t^2 \) elements. If the order of \( \overline{A} \) is \( r \), we say that \( A \) is a \((t, r)\) PH-plane. We are concerned with the problem of determining the spectrum \( S \) of all admissible pairs \((t, r)\). Since the finite projective planes are simply the \((1, r)\) PH-planes, our concern is with a generalization of the classical existence question for projective planes.

Prior to this announcement, the only pairs \((t, r)\) known to belong to \( S \) satisfy the requirements:

1. \( t \) is a power of \( r \),
2. \( r \) is a prime power.

Conversely, all such pairs do belong to \( S \), and all arise as the invariants of the Desarguesian-Pappian PH-planes investigated by Klingenberg [5]. A deep theorem of Artmann [1] allows one to assert that \((t, r)\) is in \( S \) if (1) holds and if \( r \) is the order of a projective plane. Whether this is any improvement over the previous result is, however, still uncertain.

Nonexistence results to date are also few in number. The celebrated Bruck-Ryser Theorem gives infinitely many values of \( r \) for which \((1, r) \notin S \). Clearly \((1, r) \notin S \) implies \((t, r) \notin S \) for any \( t \). Kleinfield [4] has observed that \((t, r) \in S \) with \( t \neq 1 \) implies \( t \geq r \). Most recently, Drake [2] has proved that \((t, r) \in S \) with \( 1 \neq t \neq r \) implies that \( t = 4 \) or \( 8 \) or that \( r < t + 1 - \sqrt{2t + 3} \).

The current note is written to announce the following two existence results:


Key words and phrases. Projective Hjelmslev plane, projective plane, permutation and incidence matrices, Desarguesian, Pappian.

Supported by NSF Grant GP-39059, a University of Florida Faculty Development Grant and an Alexander von Humboldt Dozentenstipendium.

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**Theorem 1.** Let \( t, r, q, b \) be positive integers such that \((t, r) \in S\), \( q \) is a prime power and \( q + 1 = t(r + 1) \). Then \((t \cdot q^b, r) \in S\).

**Theorem 2.** Let \( t, q, b \) be positive integers such that \((t, t) \in S\), \( q \) is a prime power and \( 2(t + 1) \leq q + 1 \leq t(t + 1) \). Then \((t \cdot q^b, t) \in S\).

Details will be given elsewhere of a construction which simultaneously yields both theorems and a little more. Theorem 1, for example, allows one to conclude that \((8 \cdot 23^b, 2) \in S\) for arbitrary \( b \). The actual Lenz-Drake construction applied to the "extremal" \((8, 2)\) PH-planes of Shult and Drake [3] yields the additional information that \((8 \cdot 19^b, 2), (8 \cdot 17^b, 2) \in S\).

We remark that Theorem 1 may be applied either recursively or in tandem with Theorem 2. For example, since \((2, 2) \in S\), Theorem 1 (or 2) yields \((2 \cdot 5^b, 2) \in S\). A second application of Theorem 1 then yields \((2 \cdot 5 \cdot 29^d, 2), (2 \cdot 25 \cdot 149^d, 2) \in S\).

The construction is largely elementary. We mention several of the basic ideas, presenting them in generality sufficient for the proof of Theorem 1. If \( M = [m_{ij}] \) is an incidence matrix for a \((t, r)\) PH-plane \( A \), then every row and every column of \( M \) contains precisely \( t(r + 1) \) one's. Thus König's Lemma implies that \( M \) is a sum of permutation matrices; consequently, it is possible to obtain a matrix \( N = [n_{ij}] \) of the same size as \( M \) such that \( n_{ij} = 0 \) precisely when \( m_{ij} = 0 \) and so that every integer from 1 to \( t(r + 1) \) appears in each row and each column of \( N \). Next one seeks a suitable set of \( t(r + 1) \) square matrices \( B_1, B_2, \ldots \) of order \( s^2 \) where \( s = q^b \). One then obtains a matrix \( G \) from \( N \) by substituting \( B_i \) for \( i \) when \( i \geq 1 \) and replacing each 0 by the square zero matrix of order \( s^2 \). For \( G \) to represent the desired \((t \cdot s, r)\) PH-plane, it suffices to demand that the matrices \( B_i \) satisfy:

\[(3) \quad B_i \cdot (B_j)^T = (B_j)^T \cdot B_i = J \quad \text{when } i \neq j\]

and

\[(4) \quad \sum B_i \cdot (B_i)^T, \quad \sum (B_i)^T \cdot B_i \geq 2J;\]

here \( J \) denotes the matrix of all one's, and one writes \([x_{ij}] \succeq [y_{ij}]\) to mean that \( x_{ij} \geq y_{ij} \) for all \( i, j \).

The \( B_i \) can be obtained from an \((s, q)\) PH-plane \( A' \) of the type investigated by Klingenberg and mentioned above. One obtains an incidence matrix \( D = [D_{ij}] \) for \( A' \) so written that every \( D_{ij} \) is square of order \( s^2 \) and successive sets of \( s^2 \) columns (rows) represent neighbor classes of lines (points). Let \( E_1, E_2, \ldots \) be the nonzero matrices among the \( D_{1x}; F_1, F_2, \ldots \) be the nonzero matrices among the \( D_{x1} \). Then there exist permutation matrices \( P_i, Q_i \) such that \( E_i P_i = Q_i F_i = B_i \) for all \( i \), and these \( B_i \) satisfy conditions (3) and (4).
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