PARTIAL AND COMPLETE CYCLIC ORDERS

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We show that, in contrast to a famous theorem on linear orders, not every partial cyclic order on $M = \{1, \ldots, m\}$ can be extended to a complete cyclic order. In fact, the complexity, in a certain sense, of sufficient conditions for such an extendability increases rapidly with $m$.

**Definition 1.** (i) Two linear orders, $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$, on $M$ are called cyclically equivalent if there exists $k \in M$ such that $[j - 1 \equiv (i - 1 + k) \pmod{m}] \Rightarrow a_i = b_j$.

(ii) A complete cyclic order (CCO) on $M$ is an equivalence class $C$ of linear orders modulo cyclic equivalence; denote $a_1 a_2 \cdots a_m$ for the equivalence class containing $(a_1, a_2, \ldots, a_m)$.

**Definition 2.** A partial cyclic order (PCO) on $M$ is a set $\Delta$ of cyclically ordered triples (COTs) out of $M$ such that:

(i) $xyz \in \Delta \Rightarrow zyx \notin \Delta$ ("antisymmetry"),

(ii) $\{xyz, xzw\} \subseteq \Delta \Rightarrow xyw \in \Delta$ ("transitivity"); since $xyz = zyx$, etc., also $yzw \in \Delta$ is implied.

**Theorem 3.** (i) If $C$ is a CCO then the set $\Delta$ of all COTs derived from $C$ is a PCO.

(ii) If $\Delta$ is a saturated PCO, i.e., $\{x, y, z\} \subseteq (M^3) \& xyz \notin \Delta \Rightarrow zyx \in \Delta$, then there exists a CCO from which all of $\Delta$'s COTs are derived; $\Delta$ is then said to be extendable to a CCO.

**Corollary 4.** A PCO is extendable to a CCO if and only if it is contained in a saturated PCO.

It is natural to ask whether every PCO is extendable to a CCO (or, equivalently, is contained in a saturated PCO). In view of the following example, the answer is in the negative.

**Example 5.** Let $M = \{a, b, \ldots, m\}$ be the set of the first thirteen letters, and let $\Delta = \{acd, bde, cef, dfg, egh, fha, gac, hcb, abi, ci\,j, ik\,l, jm, kma, lab, mbc, hcm, bhm\}$. Obviously, $\Delta$ is a PCO. Suppose that $\Delta^* \supset \Delta$ is a saturated PCO. If $abc \in \Delta^*$ then, since $acd \in \Delta^*$, also $bcd \in \Delta^*$. Then, also $cde \in \Delta^*$, and successive applications of transivity finally yield $acb \in \Delta^*$, which contra-
dicts antisymmetry. Thus, $abc \notin \Delta^*$ and, therefore, by saturatedness, $acb \in \Delta^*$. Analogously, since $abi \in \Delta^*$, also $cbi \in \Delta^*$, and successive applications of transitivity finally yield $abc \in \Delta^*$. Thus, antisymmetry is contradicted again. It follows that there is no saturated PCO that contains $\Delta$.

The unextendability of $\Delta$ in Example 5 followed essentially from the fact that neither $abc$ nor $cba$ belongs to any PCO that contains $\Delta$. That gives rise to the following definition.

**Definition 6.** If $r = \{i, j, k\}$, $1 \leq i < j < k \leq m$, denote $\tau^+ = ijk$ and $\tau^- = kji$ for the two possible cyclic orderings of $\tau$. A PCO $\Delta$ is said to satisfy the $n$th order condition if for every $\tau_1, \ldots, \tau_n \in \binom{M}{3}$ there exists a PCO $\Delta^* \supset \Delta$ and $e_i \in \{+,-\}$ ($i = 1, \ldots, n$), such that $\{\tau_1^{e_1}, \ldots, \tau_n^{e_n}\} \subset \Delta^*$.

Obviously, all the $n$th order conditions ($n = 0, 1, \ldots$) are necessary for extendability to a CCO and, as $n$ increases, the $n$th order condition becomes stronger. The conjunction of all the $n$th order conditions ($n = 0, 1, \ldots$) is a sufficient condition for every $m$. It is natural to ask whether there exists an $n$ such that the $n$th order condition suffices for every PCO $\Delta$ on a finite set $M$ to be extendable at a CCO. Unfortunately, the answer to this question also is in the negative. A sequence of PCOs that prove this is constructed as follows.

**Example 7.** Let $m_0 = 13$ and let $\Delta_0$ be the PCO on $M_0 = \{1, \ldots, 13\}$ defined in Example 5 (identify $a$ with 1, $b$ with 2, etc.). As we have already seen, $\Delta_0$ is not extendable to a CCO. However, since it is a PCO, it satisfies the 0th order condition. For the purpose of later use in induction, note that $\Delta_0 \setminus \{egh\}$ is extendable to the following complete cyclic ordering: $abhchgdeijklm$. Suppose, by induction, that $\Delta_n$ is a PCO on $M_n = \{1, \ldots, m_n\}$, $\Delta_n$ satisfies the $n$th order condition but is not extendable to a CCO. Suppose also that $xyz \in \Delta_n$ is such that $\Delta_n \setminus \{xyz\}$ is extendable to a CCO. We construct $\Delta_{n+1}$ as follows. Define $m_{n+1} = m_n + 15$ and $M_{n+1} = \{1, \ldots, m_{n+1}\}$ and denote $(u_1, \ldots, u_5, v_1, \ldots, v_5, w_1, \ldots, w_5) = (m_n + 1, \ldots, m_{n+1})$. Let

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\Delta' = (\Delta_n \setminus \{xyz\}) \cup \{zu_1u_2, yu_2u_3, u_1u_3u_4, u_2u_4u_5, u_3u_5x, u_4xy, xyv_1, u_1v_1v_2, yv_2v_3, v_1v_3v_4, v_2v_4v_5, u_3v_5x, v_4xy, v_5yu_1, u_5yw_1, zw_1w_2, yw_2w_3, w_1w_3w_4, w_2w_4w_5, w_3w_5u_5, w_4u_5y, w_5yz\}.
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Let $\Delta_{n+1}$ be the transitive closure of $\Delta'$, i.e., $\Delta_{n+1}$ is the intersection of all the transitive classes of COTs that contain $\Delta'$ (see Definition 2). It turns out that $\Delta_{n+1}$ is a PCO, but is not extendable to a CCO. Also, $\Delta_{n+1} \setminus \{w_3w_5u_5\}$ is extendable to a CCO. The proof of these facts follows from the analogous properties of $\Delta_n$. The important property of $\Delta_{n+1}$ is that it satisfies the $(n+1)$st order condition. A detailed proof will be given elsewhere. Here, we indicate that two cases are distinguished when a set of 3-element subsets of $M_{n+1}$ is given.
First, when \( |\tau_i \cap M_{n+1}| \geq 2 \) for \( i = 1, \ldots, n + 1 \), the \( e_i - s \) are determined essentially by the CCO on \( M_n \) to which \( \Delta_n \setminus \{xyz\} \) is extendible. Otherwise, the induction hypothesis is applied and the \( e_i - s \) are determined essentially by a PCO \( \Delta^* \) that contains \( \Delta_n \) and \( n \) of the \( \tau_i - s \).

In view of Example 7, an algorithm for extending a PCO to a CCO which is based on successive addings of COTs, cannot be polynomial. We conjecture that there is no polynomial algorithm for this problem; note that there seems to be an equivalence between our problem and that of the Hamiltonian path, from the point of view of complexity of computations.