FIXED POINTS OF DISK ACTIONS

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Communicated October 30, 1975

As a sequel to a previous announcement [3], the author can now give a complete classification up to homotopy type of which spaces can occur as fixed point sets of smooth actions of a given compact Lie group on disks. The result is contained in Theorems 1 to 3 below. For a group $G$, $G_0$ denotes its identity component.

**Theorem 1.** Let $G$ be a compact Lie group, and $F$ a finite CW complex. Then there exists a smooth action of $G$ on a disk with fixed point set having the homotopy type of $F$ if and only if:

1. $G \cong T^n$ ($n \geq 1$): $F$ is $\mathbb{Z}$-acyclic;
2. $G_0$ a torus and $|G/G_0| = p^a$ ($p$ prime, $a \geq 1$): $F$ is $\mathbb{Z}_p$-acyclic,
3. $G_0$ not a torus or $G/G_0$ not of prime power order: $\chi(F) \equiv 1 \pmod{n_G}$

for some fixed integer $n_G$.

In order to describe the calculations of $n_G$, some classes of finite groups are defined, as in [3] and [4]. $G^1$ denotes the class of all $G$ with normal subgroup $P$ of prime power order, such that $G/P$ is cyclic. For $q$ prime, $G^q$ denotes the class of all $G$ with normal subgroup $H \in G^1$ of $q$-power index. Then one gets

**Theorem 2.** 1. If $G_0$ is not a torus, then $n_G = 1$.
2. If $G_0$ is a torus, then $n_G = n_{G/G_0}$.
3. If $G$ is finite, then $n_G = 0$ if and only if $G \in G^1$; if $G \notin G^1$ then for any prime $q$, $q \nmid n_G$ if and only if $G \in G^q$.

In Theorem 1, the necessity of the conditions in (1) and (2) follow from standard Smith theory. Sufficiency follows in (2) from Jones [2], and in (1) is trivial ($G \ast F$ is contractible and can be thickened up to a disk action by Theorem 6 of [4]).

For finite $G$, the existence of $n_G$ and the calculations in Theorem 2, part 3, were proven in [4]. Furthermore, if $G_0$ is a torus and $G \supseteq G_0$, then $F$ clearly has the homotopy type of the fixed point set of a disk action of $G$ if and only if it does the same for $G/G_0$, so $n_G = n_{G/G_0}$. The case where $G_0$ is nontoral will be dealt with below; the above theorems say that any finite homotopy type can occur as fixed point set for such $G$.

The following result, completing the calculation of $n_G$, was obtained in


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Theorem 4 of [5] by studying the projective obstruction $\gamma_G(F)$ first introduced in [4].

**Theorem 3.** For any finite group $G$, $n_G = 4$ if and only if:

1. $G$ is a semidirect product $0 \rightarrow \mathbb{Z}_n \rightarrow G \rightarrow \mathbb{Z}_2^k \rightarrow 0$ ($n$ odd) given by an automorphism $\alpha \in \text{Aut}(\mathbb{Z}_n)$.
2. $G \notin G^1$, but the subgroup of index 2 is in $G^1$.
3. Letting $\alpha$ also denote the induced automorphism of $\mathbb{Z}_n^k$ (the ring generated by the $n$th roots of unity), there is no unit $u \in (\mathbb{Z}_n^k)^*$ such that $\alpha(u) = -u$. Otherwise, $n_G$ equals 0, 1 or a product of distinct primes.

Groups fulfilling conditions 1–3 do actually exist, the smallest being given by $\langle a, b : a^{15} = b^4 = e, bab^{-1} = a^2 \rangle$.

It remains to describe the case of groups with nontoral identity component; by Bredon’s construction [1, §1.8] it is enough to construct a fixed point free action of any such group on a disk. The following theorem provides some very specific examples of such actions. The concept of a family of subgroups is used, as defined by tom Dieck.

**Theorem 4.** Let $G$ be a compact Lie group, and $\mathcal{F}$ a nonempty family of subgroups. Then there exists a smooth action of $G$ on a disk $D$ such that $D^H$ is a disk for $H \in \mathcal{F}$ and empty for $H \notin \mathcal{F}$, if and only if:

1. For any pair of subgroups $H < K$ in $G$, for which $K/H$ has prime order, either both $H$ and $K$ are in $\mathcal{F}$ or neither is.
2. $\mathcal{F}$ is closed in the space of closed subgroups of $G$ with the Hausdorff topology.

In particular, the family of subgroups $H$ such that $H_0$ is a torus and $H/H_0$ solvable meets these conditions.

**REFERENCES**

5. ————, *Projective obstructions to group actions on disks* (to appear)