ON THE TAMAGAWA NUMBER OF QUASI-SPLIT GROUPS

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1. Introduction. In this paper we give a formula for the Tamagawa number $\tau(G)$ (see [6]) of a connected semisimple quasi-split algebraic group $G$ defined over an algebraic number field $F$. The method used is that of R. P. Langlands (see [2]).

Let $A$ be the adeles of $F$; $G_A$ the locally compact adele group of $G$ in which the group $G_F$ of $F$-rational points is embedded.

Let $B$ be the Borel subgroup of $G$ defined over $F$, and $A$ the maximal torus of $B$ defined over $F$. $\tau(A)$ is the Tamagawa number of $A$. $L_F$ (resp. $L_F^+$) denotes the lattice of $F$-rational weights of $G$ (resp. of the simply-connected form of $G$). Let $c$ be the index $[L_F^+: L_F]$. Then the main formula is

THEOREM. $T(G) = c\tau(A)$.

2. Sketch of the proof. Let $P$ be the orthogonal projection of $L^2(G_F \setminus G_A)$ onto the space of constant functions. Langlands [2] observes the simple relation:

\[(1, 1)(P\varphi^\sim, P\psi^\sim) = (\varphi^\sim, 1)(1, \psi^\sim)\]

where $\varphi^\sim, \psi^\sim \in L^2(G_F \setminus G_A)$ and $(\cdot, \cdot)$ is the inner product on $L^2(G_F \setminus G_A)$.

As

\[(1, 1) = \int_{G_F \setminus G_A} dg,\]

the problem reduces to the computation of the remaining three terms in (1).

Let $G_v = \Pi_{v|\infty} G_{F_v}$ where $F_v$ is the completion of $F$ at the place $v$ and \(v|\infty\) means that $v$ is infinite. Let $K_\infty$ be the maximal compact subgroup of $G_\infty$, and $K_0 = \Pi_{v<\infty} G_{O_v}$ where \(v<\infty\) means that $v$ is finite, $O_v$ is the maximal compact subring of $F_v$ and $G_{O_v}$ is the compact subgroup of $G_{F_v}$ consisting of elements with coefficients in $O_v$ and whose determinants are units. Put

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This paper is based on the author's Ph. D. dissertation, written at Yale University under Professor G. D. Mostow. The problem and the approach were suggested by R. P. Langlands.
$K = K_\infty \cdot K_0$. Then there exists a finite set $\{g_i \in G_A | 1 \leq i \leq n\}$ such that

$$G_A = \bigcup_{i=1}^n B_A g_i K.$$

Let $N$ be the unipotent radical of $B$, pick continuous functions $\varphi, \psi$ defined on $N_A B_F \setminus G_A / K$ such that we have a Fourier integral expression

$$\varphi(g) = \int_{|\lambda|=\lambda_0} \Phi^\lambda(g) \, d\lambda$$

for a suitable quasi-character $\lambda_0$ of $A_F \setminus A_A$ and the series

$$\varphi^\sim(g) = \sum_{\gamma \in B_F \setminus G_F} \varphi(\gamma g)$$

converges to an element in $L^2(G_F \setminus G_A)$. Similarly, we have

$$\psi(g) = \int_{|\lambda|=\lambda_0} \Psi^\lambda(g) \, d\lambda.$$

The $\Phi, \Psi$ are functions in $\lambda$ and $g$, and there exists a sesquilinear pairing $\langle \cdot, \cdot \rangle$ between these functions such that

$$\langle \varphi, 1 \rangle = \langle \Phi^\rho, 1 \rangle, \quad (1, \psi) = \langle 1, \Psi^\rho \rangle$$

where $\rho$ is the half sum of the positive roots of $G$.

To evaluate the remaining terms $(P\varphi, P\psi)$, we introduce an unbounded self-adjoint operator $A$ on the closed subspace $L$ of $L^2(G_F \setminus G_A)$ generated by the functions $\varphi^\sim$ with $\varphi$ of the form indicated above. If $\mathbb{E}(x)$ is a right continuous spectral resolution of $A$, then we have

$$P = \mathbb{E}((\rho, \rho)) - \mathbb{E}(\rho, \rho) - 0,$$

$$\langle P\varphi^\sim, P\psi^\sim \rangle = \frac{1}{ct(A)} \lim_{s \to 1} \frac{\langle M(w, \rho^s) \Phi^\rho^s, \Psi^{\rho^s}_{w^\sim} \rangle}{L(s, A)},$$

where $w$ is the element of the Weyl group that sends every positive root to negative root, $s$ is a complex number, $L(s, A)$ is the $L$-function of $A$ (see [4], [5]) and $M(w, \rho^s)$ is a linear map on a vector space of functions on $N_A B_F \setminus G_A / K$.

There exists a finite set $S$ of places of $F$ such that

$$M(w, \rho^s) \Phi^\rho^s(g) = \left( \prod_{v \notin S} \int_{N_{F_v}} \Phi^\rho^s(wn_v) \, dn_v \right) \left( \int_{N_S} \Phi^\rho^s(wn_sg_s) \, dn_s \right),$$

where $g = (g_v) \in G_A$ is such that $g_v = 1$ if $v \notin S$, $n_s \in N_S = \Pi_{v \in S} N_{F_v}$.

Let $\overline{N}$ be the unipotent radical of the Borel subgroup opposite to $B$. Write $\overline{N}^w = w^{-1} N_w \cap \overline{N}$. Then we have

$$\int_{\overline{N}^w_{F_v}} \Phi^\lambda(n) \, d\overline{n} = \frac{\det(I - |\omega| \sigma \text{Ad} \hat{f}|_{\hat{w}})}{\det(I - \sigma \text{Ad} \hat{f}|_{\hat{w}})}.$$
where $\Phi^A(1) = 1$ (for notation see [3], [4]). Formula (10) is proved first for all rational rank one quasi-split groups by explicit computation and then for the general case by the method of Bhanu-Murti, Gindikin and Karpelevic [1]. From (10) we get

$$\lim_{s \to 1} \prod_{v \in S} \int_{N_{F_v}} \Phi^S(w_{n_v}) \ dn_v = \left( \lim_{s \to 1} \prod_{v \in S} L_v(s, A) \right) \left( \prod_{v \in S} \text{volume } G_0 v \right).$$

The remaining integral in (9) is calculated by comparing the decomposition of the measure on $G_A$ according to the Iwasawa decomposition and the Bruhat decomposition. We get

$$\lim_{s \to 1} \int_{N_S} \Phi^S(w g_S) \ dn_S = \frac{\langle \Phi^A, 1 \rangle \Pi_{v \in S} L_v(1, A)}{\Pi_{v \in S} \text{volume } G_0 v}.$$  

The theorem now follows immediately from (1), (2), (6), (8)–(12).

It follows from our theorem that Weil's conjecture on Tamagawa is true for quasi-split group.

REFERENCES


