ON SCARCITY OF OPERATORS WITH FINITE SPECTRUM
BY BERNARD AUPETIT

Communicated by Paul R. Halmos, January 9, 1976

Let \( p \) denote the spectral radius of an operator; in 1968–1970 Edoardo Vesentini proved

**Proposition 1 (\[5\] and \[6\]).** If \( \lambda \rightarrow f(\lambda) \) is an analytic function mapping a domain in \( \mathbb{C} \) into a complex Banach algebra then \( \lambda \rightarrow \log p(f(\lambda)) \) is subharmonic.

From this we got the following generalization of Newburgh's continuity theorem [4], where \( o(x) \) is the union of \( \text{Sp } x \) and its holes.

**Proposition 2 (Almost-Continuity Theorem).** If \( \lambda \rightarrow f(\lambda) \) is analytic on a domain \( D \) containing \( z_0 \) and if \( E \) is a subset of \( D \), such that \( z_0 \in \bar{E} \), \( E \) is nonsharp at \( z_0 \), then there exists a sequence \( (\lambda_n) \) converging to \( z_0 \) with \( \lambda_n \in E \), \( \lambda_n \neq z_0 \), and \( \lim_{n \to \infty} o(f(\lambda_n)) = o(f(z_0)) \).

The same statement with the spectrum is false. If \( \delta \) is the diameter of the spectrum, we obtained as well

**Proposition 3.** With the same hypothesis \( \lambda \rightarrow \log \delta(f(\lambda)) \) is subharmonic.

All of these results and intricate properties of subharmonic functions, capacity and sharp sets, easily found in [1], give the fundamental theorem

**Theorem 1.** Either the set of \( \lambda \), such that the spectrum of \( f(\lambda) \) is finite, is of outer capacity zero, or there exists an integer \( n \) such that the spectrum of \( f(\lambda) \) has exactly \( n \) elements, for every \( \lambda \), except on a closed set of capacity zero, where the spectrum has at most \( n - 1 \) elements.

Kaplansky [3], in 1954, and Hirschfeld-Johnson [2], in 1972, proved that \( A/\text{Rad } A \) is finite dimensional, for a complex Banach algebra \( A \), if the spectrum of every element of this algebra is finite. Unfortunately the method does not work for local and real cases. Other persons (Behncke, Wong) obtained the same result for \( A^* \)-algebras supposing the spectrum finite for hermitian elements.

Theorem 1 can be used to get


1 The author is supported by National Research Council of Canada (Grant A7668).
Theorem 2. Let $A$ be a complex Banach algebra, $H$ a closed real subspace of $A$ such that $A = H + iH$. If there exists a nonempty open set $U$ of $H$ such that $x \in U$ implies $\text{Sp } x$ finite, then $A/\text{Rad } A$ is finite dimensional.

And particularly

Theorem 3. Let $A$ be a real Banach algebra containing a nonempty open set $U$ such that $x \in U$ implies $\text{Sp } x$ finite; then $A/\text{Rad } A$ is finite dimensional.

Theorem 4. Let $A$ be a complex Banach algebra with involution such that the set of hermitian elements contains a nonempty open set $U$ with the property $x \in U$ implies $\text{Sp } x$ finite; then $A/\text{Rad } A$ is finite dimensional.

Full details will appear elsewhere.

REFERENCES


DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ LAVAL, QUÉBEC, CANADA