MAYER-VIETORIS SEQUENCES FOR COMPLEXES
OF DIFFERENTIAL OPERATORS

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Communicated by I. M. Singer, January 30, 1976

This is an announcement of some of the results in \[1\].

1. Preliminaries. Let \(X\) be a smooth manifold, \(E^i, i = 0, 1, \ldots\), smooth vector bundles, and \(\Omega \subset X\) open. Let \(E^i(\Omega) = C^\infty(\Omega, E^i)\). We consider complexes of linear differential operators with locally constant orders

\[
E(\Omega): E^0(\Omega) \xrightarrow{D^0} E^1(\Omega) \xrightarrow{D^1} \cdots.
\]

The cohomology of \(E(\Omega)\) is \(H^1(\Omega) = \ker D^1/\text{im } D^0\). Let \(S \subset \Omega\) be a smooth hypersurface dividing \(\Omega\) into two parts: \(\Omega - S = \hat{\Omega}^+ \cup \hat{\Omega}^-; \hat{\Omega}^+ \cap \hat{\Omega}^- = \emptyset\); and \(S \cup \hat{\Omega}^\pm = \Omega^\pm\). Let \(E^i(\Omega^\pm)\) be the sections over \(\Omega^\pm\) smooth up to \(S\). We obtain

\[
E(\Omega^\pm): E^0(\Omega^\pm) \xrightarrow{D^0} E^1(\Omega^\pm) \xrightarrow{D^1} \cdots.
\]

A section \(u \in E^i(\Omega)\) has zero Cauchy data on \(S\) if \(D^i\tilde{u} = \tilde{f}\) is valid on \(\Omega\) in the sense of distributions where \(\tilde{u} = u\) on \(\Omega^+\) and \(= 0\) on \(\Omega - \Omega^+\), and \(\tilde{f} = D^i u\) on \(\Omega^+\) and \(= 0\) on \(\Omega - \Omega^+\); and similarly with \(\Omega^+\) replaced by \(\Omega^-\). The space of such sections is \(I(\Omega, S)\), and \(I(\Omega^\pm, S) = I(\Omega, S)|_{\Omega^\pm}\). We obtain complexes

\[
I(\Omega, S): I^0(\Omega, S) \xrightarrow{D^0} I^1(\Omega, S) \xrightarrow{D^1} \cdots,
\]

and

\[
I(\Omega^\pm, S): I^0(\Omega^\pm, S) \xrightarrow{D^0} I^1(\Omega^\pm, S) \xrightarrow{D^1} \cdots,
\]

with cohomologies \(H^i(\Omega, I)\) and \(H^i(\Omega^\pm, I)\), respectively.

The tangential complex is the quotient complex \(0 \to I(\Omega, S)\to E(\Omega)\to C(S)\to 0\). An element of \(C^i(S)\) is Cauchy data for \(D^i\), the induced operator is \(D^i\), and the cohomology is \(H^i(S)\).
Let \( F(\Omega, S) \) (resp. \( F(\Omega^\pm, S) \)) be the space of sections of \( E(\Omega) \) (resp. \( E(\Omega^\pm) \)) which vanish to infinite order on \( S \). The quotient \( E/F(S) \) may be thought of as sequences of sections over \( S \) representing the normal derivatives of all orders of sections in \( E(\Omega) \). The diagram

\[
\begin{array}{ccccccccc}
0 & 0 & & & & & & & \\
& & & & & & & & \\
& & 0 & C(S) & C/F & 0 & & & \\
& & & & & & & & \\
0 & \rightarrow & F(\Omega) & \rightarrow & E(\Omega) & \rightarrow & E/F & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
0 & \rightarrow & F(\Omega) & \rightarrow & I(\Omega) & \rightarrow & I/F & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
0 & 0 & 0 & 0 & & & & & \\
\end{array}
\]

(1)

commutes and has exact rows and columns.

The cohomology of

\[
\begin{array}{cccc}
0 & \rightarrow & (I/F)_0 & D_0 & (I/F)_1 & D_1 & \cdots, \\
\end{array}
\]

denoted by \( H^i(I/F) \), is the Cauchy-Kowalewski cohomology of \( E(\Omega) \) on \( S \).

**Definition.** \( S \) is formally noncharacteristic for \( E \) if \( H^i(I/F) = 0 \), all \( i \geq 0 \).

2. The Mayer-Vietoris sequence in cohomology.

**Theorem 1.** If \( S \) is formally noncharacteristic for \( E \), then

\[
0 \rightarrow H^0(\Omega) \rightarrow H^0(\Omega^+) \oplus H^0(\Omega^-) \rightarrow H^0(S) \rightarrow H^1(\Omega) \rightarrow \cdots
\]

is an exact sequence.

**Proof.** By the Whitney extension theorem

\[
0 \rightarrow E(\Omega) \rightarrow E(\Omega^+) \oplus E(\Omega^-) \rightarrow E/F \rightarrow 0
\]

is an exact sequence, where the first map is restriction and the second is the jump in normal derivatives. The long exact sequence of this gives (3) with \( H^i(E/F) \) instead of \( H^i(S) \). But the long exact sequence of the last column of (1), with \( H^i(I/F) = 0 \), gives \( H^i(E/F) \cong H^i(S) \).

The proof of the following is in the same spirit.

**Theorem 2.** If \( S \) is formally noncharacteristic for \( E \), then
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0 \rightarrow H^0(\Omega^+, I) \rightarrow H^0(\Omega) \rightarrow H^0(\Omega^-) \rightarrow H^1(\Omega^+, I) \rightarrow \cdots

\rightarrow H^0(\Omega^+ \oplus \Omega^-) \rightarrow H_0(S) \rightarrow H_1(\Omega^+, I) \rightarrow \cdots

\text{commutes, and has exact rows.}

\text{DEFINITION. A cotangent vector } \xi \in T^*_x(X) \text{ is } \text{noncharacteristic} \text{ for } E \text{ if the principal symbol complex}

\[ 0 \rightarrow E^0_x \xrightarrow{\sigma_\xi(D^0)} E^1_x \xrightarrow{\sigma_\xi(D^1)} \cdots \]

\text{is exact. The surface } S \text{ is noncharacteristic if the normal cotangent vector field is noncharacteristic at each point.}

\text{THEOREM 3. If } S \text{ is noncharacteristic for } E, \text{ then } S \text{ is formally noncharacteristic for } E.

\text{This is a formal version of the Cauchy-Kowalewski theorem for complexes of operators.}

\text{If } E(\Omega) \text{ and } E(\Omega^\pm) \text{ are given the topology of uniform convergence of all derivatives on compact sets, and if subspaces and quotients are given the induced and quotient topologies, the maps in Theorems 1 and 2 are continuous.}

\text{REMARK. We may consider the spaces of sections with support in any regular paracompactifying family of supports, in particular compact supports, topologized as the strict inductive limits of Fréchet-Schwartz spaces, and all the above results still hold.}

3. \textbf{The Mayer-Vietoris sequence in homology.} Consider the topological duals of the above complexes.

Denote the homology of

\[ 0 \leftarrow (E^0(\Omega))' \xleftarrow{(D^0)'} (E^1(\Omega))' \xleftarrow{(D^1)'} \cdots \]

by } H_1(\Omega), \text{ and similarly for the other complexes.

\text{THEOREM 4. If } S \text{ is formally noncharacteristic,}

\[ 0 \leftarrow H_0(\Omega) \leftarrow H_0(\Omega^+) \oplus H_0(\Omega^-) \leftarrow H_0(S) \leftarrow H_1(\Omega) \leftarrow \cdots \]

\text{is exact, and}

\[ 0 \leftarrow H_0(\Omega^+, I) \leftarrow H_0(\Omega) \leftarrow H_0(\Omega^-) \leftarrow H_1(\Omega^+, I) \leftarrow \cdots \]

\text{commutes and has exact rows.}
The open mapping theorem for Fréchet-Schwartz spaces shows that if $S$ is formally noncharacteristic, the dual of (2) is exact. The rest of the proof is as in Theorems 1 and 2.

The proof fails in the case of sections with supports in a regular paracompactifying family of supports. However, if $S$ is noncharacteristic, the proof of Theorem 3 can be adapted to show the dual of (2) is exact, and the theorem analogous to Theorem 4 for sections with restricted supports holds. Similar methods give the analogues of Theorems 1 and 2 for distributions. One also obtains duality theorems between homology and cohomology.

REFERENCES

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