MARKOV PROCESSES ON MANIFOLDS OF MAPS

BY PETER BAXENDALE

Communicated by Daniel W. Stroock, January 2, 1976

1. Introduction. In this note we describe a construction of a Markov process on a manifold of maps starting from a Gaussian measure on the space of sections of an associated vector bundle. Let $S$ be a compact metric space of finite metric dimension and $M$ a smooth complete finite dimensional Riemannian manifold. Our basic construction gives a family $\{\nu_t; t \geq 0\}$ of Borel probability measures on the space $C(S \times M, M)$ of continuous functions from $S \times M$ to $M$ with the compact-open topology. The multiplication $(f, g)(s, m) = f(s, g(s, m))$ for $f, g \in C(S \times M, M)$ makes $C(S \times M, M)$ into a topological semigroup with identity. Then $\nu_t \ast \nu_s = \nu_{t+s}$ for $s, t \geq 0$ and the right translates of the $\nu_t$ give transition probabilities for a Markov process on $C(S \times M, M)$ with continuous sample paths. The left action of $C(S \times M, M)$ on $C(S, M)$ induces a Markov process on $C(S, M)$ with transition probability $\nu_{t,s} = \text{image of } \nu_t$ under the action of $C(S \times M, M)$ on $g \in C(S, M)$.

2. Statement of results. Let $\xi$ denote the product bundle $S \times TM \to S \times M$ and $C(\xi)$ the space of continuous sections of $\xi$. Given a Gaussian measure $\mu$ of mean zero on $C(\xi)$, define

$$Q(s, x, t, y) = \int f(s, x) \otimes f(t, y) \, d\mu(f) \in T_xM \otimes T_yM$$

for all $s, t \in S, x, y \in M$.

$Q$ is a reproducing kernel for the bundle $\xi$ (see Baxendale [1]) and determines $\mu$ uniquely. Let $X \in C(\xi)$.

For a closed isometric embedding of $M$ inside some Euclidean space $V$, let $h(x)$ denote the second fundamental form for $M \subset V$ at $x \in M$. Using the natural inclusion $T_xM \subset V$ and orthogonal projection $V \to T_xM$, we think of $X, Q$ and $h$ taking values in $V$ and its various tensor products.

**Theorem 1.** Suppose there exists a closed isometric embedding $M \subset V$ such that (i) $h$ is bounded and uniformly Lipschitz with respect to the metric on $M$ induced from $V$.

Suppose moreover that there exist a Gaussian measure $\mu$ on $C(\xi)$, $X \in C(\xi)$ and $\alpha > 0, C > 0$ such that

\[
\text{tr}(Q(s, x, s, x)) \leq C, \quad \forall s, x,
\]
\[
\text{tr}(Q(s, x, s, x) + Q(t, y, t, y) - Q(s, x, t, y) - Q(t, y, s, x)) \leq C(d(s, t)^2 + |x - y|^2), \quad \forall s, x, t, y,
\]
\[
|X(s, x)|_V \leq C, \quad \forall s, x,
\]
\[
|X(s, x) - X(t, y)|_V \leq C(d(s, t)^{\alpha} + |x - y|_V), \quad \forall s, x, t, y.
\]

Then \(\mu\) and \(X\) determine a family of Borel probability measures \(\{v_t: t \geq 0\}\) on \(C(S \times M, M)\) satisfying

(a) \(v_s \ast v_t = v_{s+t}, \quad \forall s, t \geq 0\),

(b) the \(v_t\) are transition probabilities for a Markov process on \(C(S \times M, M)\) with continuous sample paths.

We illustrate the dependence of the \(\{v_t\}\) on \(\mu\) and \(X\) as follows. For \(s = (s_1, \ldots, s_r) \in S^r\) and \(x = (x_1, \ldots, x_r) \in M^r\) denote by

\[
\rho_{s,x}: C(S \times M, M) \rightarrow M^r
\]
\[
\sigma_{s,x}: C(\xi) \rightarrow T_{x_1}M \times \cdots \times T_{x_r}M,
\]
the evaluation maps at \((s_1, x_1), \ldots, (s_r, x_r)\). Let \(\nu_{s,x}^t\) be the image of \(v_t\) under \(\rho_{s,x}\), then

(i) the \(\nu_{s,x}^t\) for all \(s, x\) determine \(v_t\),

(ii) the \(\nu_{s,x}^t\) for fixed \(s\) are the transition probabilities for a Markov process on \(M^r\) with continuous sample paths.

**Theorem 2.** The Markov process corresponding \(\{\nu_{s,x}^t: t \geq 0, x \in M^r\}\) has infinitesimal generator \(A_y\), where

\[
(A_y g)(x) = \frac{1}{2} \int (\nabla^2 g)(x) (\sigma_{s,x}(h), \sigma_{s,x}(h)) d\mu(h) + (\nabla g)(x)(\sigma_{s,x}(X)),
\]
where \(\nabla\) is covariant differentiation with respect to the product Riemannian structure on \(M^r\).

**3. The construction.** For each \(s, x\) and \(t \geq 0\), we construct a measure \(\nu_{s,x}^t\) on \(M^r\) as follows. Using the embedding \(M \subset V\) and choosing suitable extensions, we construct a Wiener process \(W_t\) in \(C(S \times V, V)\) (see Gross [2]) and \(X \in C(S \times V, V)\). Define

\[
Y(s, x) = \frac{1}{2} \int_{C(\xi)} h(x)(g(s, x), g(s, x)) d\mu(g) \in T^1_xM \quad \text{for} \ x \in M,
\]
and extend to \(\tilde{Y} \in C(S \times V, V)\). Consider the stochastic differential equation in \(V^r\)
\[ d\eta_i(t) = (X + \tilde{Y})(s_i, \eta_i(t)) \, dt + dW(t)(s_i, \eta_i(t)) \] for \( i = 1, \ldots, r. \)

The choice of \( \tilde{Y} \) ensures that if \( x_t \in M \), then \( \eta_i(t) \in M \) for all \( t > 0 \) with probability one. The conditions (i), (ii) and (iii), plus care in choosing extensions, imply that the equation has a solution for all \( t > 0 \), that the solution is continuous with probability one and has finite moments of all orders. We define \( \nu_{\mu}^{p, q} \) to be the distribution of \( (\eta_1(t), \ldots, \eta_r(t)) \in M^r \). The existence of the \( \{\nu_t\} \) follows from the Daniell-Kolmogorov construction and an estimate on the moments of solutions of the stochastic differential equation.

4. Examples. Suppose \( S \) and \( M \) are compact Riemannian manifolds and \( p > \frac{1}{2} \dim S, q > \frac{1}{2} \dim M + 1 \). Then \( L^2_p(S) \otimes L^2_q(TM) \subset C(\xi) \) is radonifying, and the Wiener measure \( \mu \) satisfies the conditions of Theorem 1. Take \( X = 0 \).

Each pair \( (p, q) \) in the range above yields a different family \( \{\nu_t\} \) of measures on \( C(S \times M, M) \).

The condition that \( M \) be compact may be replaced by completeness together with certain curvature conditions.

Notice that the case \( M = \mathbb{R}^n \) yields Gaussian measures. Also \( S = \text{point} \) and suitable choice of \( \mu \) gives Brownian motion on \( M \) under the sole condition that there exists a closed isometric embedding with \( \|h(x)\| \leq C(1 + d(x, x_0)) \) for some \( C > 0 \) and \( x_0 \in M \).

REFERENCES


DEPARTMENT OF MATHEMATICS, KING'S COLLEGE, ABERDEEN, SCOTLAND