INFINITE LOOP MAPS AND THE COMPLEX $J$-HOMOMORPHISM

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ABSTRACT. We study the complex $J$-homomorphism $j: U \to SG$ as the composition of two infinite loop maps.

1. Introduction. Let $p$ be an odd prime and let $q$ be a prime generating the units of $\mathbb{Z}/p^2$. All spaces will be $p$-localized. The solution of the Adams conjecture establishes a commutative diagram of fibre sequences.

$$
\cdots \to U \xrightarrow{\psi^{-1}} U \xrightarrow{\omega} BU \oplus \xrightarrow{\psi^{-1}} BU \oplus \\
\cdots \to U \xrightarrow{j} SG \xrightarrow{SG/U} BU \oplus (1.1)
$$

Several, possibly different, $\tau$ have been constructed ([2], [5] and [8]). Given $\tau$, then $\mu$ is unique. The fibre sequences are sequences of infinite loop maps and it is natural to ask whether (1.1) can be extended arbitrarily to the right—the infinite loop Adams conjecture. By [4] this would be true if $\tau$ were an infinite loop map. These results suggest strongly the validity of the conjecture.

In [2] an $H$-map, $\tau$, is given. If $F_q$ is the field with $q$ elements the finite dimensional vector spaces over $F_q$ under direct sum form a permutative category from which the infinite loopspace $J^\otimes$ is constructed by the technique of [1]. Similarly $SG$ is obtained from a category of finite sets under cartesian product. The forgetful functor gives the “discrete models” infinite loop maps $\delta: J^\otimes \to SG$.

**Theorem 1.** If $\tau$ is the map constructed in [2] then $\mu = \delta$ in (1.1).

$J^\otimes$ is the infinite loop space obtained from a category of vector spaces of $F_q$ under tensor product. Assigning to a set the vector space generated by its elements gives $\nu: SG \to J^\otimes$. Define Coker $J^\otimes$ by the infinite loop fibering 

$\xrightarrow{\pi} SG \xrightarrow{\nu} J^\otimes$.

**Theorem 2.** $\nu \circ f: J^\otimes \to J^\otimes$ is a homotopy equivalence for any map $f: J^\otimes \to SG$ such that $f_\#$ is nontrivial on $\pi_{2p-3}$.


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Theorem 3. In (1.1), $j = \delta \circ \omega$.

Combining this with Theorem 2 we easily obtain

Theorem 4 [9]. There is an equivalence of infinite loopspaces $\delta + \pi$:

$$J^\oplus \times \text{Coker } J^\oplus \rightarrow SG.$$ 

2. If the infinite loop Adams conjecture were true then there would exist an infinite loop map $J^\oplus \rightarrow SG$ satisfying Theorems 1, 3 and 4.

Theorems 2, 3 and 4 can be proved without mentioning $\tau$ at all, i.e. without the solution of the Adams conjecture. For example cf. [7, I].

Proof of Theorem 3. In [6] a cohomology theory, $Ad_q^*$ is constructed satisfying

$$[X, Z \times J^\oplus] = Ad^0_q(X)$$

and giving an infinite loopspace structure to $Z \times J^\oplus$ extending the usual one on $J^\oplus$. $Ad^0_q(X)$ has a description in terms of isomorphisms of $\mathbb{Z}/q$-vector bundles

$$\theta: E^\otimes q \rightarrow E \oplus (E' \otimes N)$$

where $E, E'$ are complex vector bundles over $X$ and $N$ is the complex regular representation of $\mathbb{Z}/q$. A similar theory constructed from isomorphisms, $\theta$, such that

$$\mu(\theta): E^{\text{diag}} \rightarrow E^\otimes \rightarrow E \oplus (E' \otimes N) \rightarrow E$$

is a proper map is also $Ad^*_q$. Sending $\theta$ to the stabilization of $\mu(\theta)$ gives an exponential $H$-map,

$$\mu: \bigcup_{n > 0} (n) \times J^\oplus \rightarrow \bigcup_{n > 0} Q_n S^0,$$

where $Q_n S^0$ is the set of maps of degree $q^n$ in $\Omega^\infty S^\infty$. It is easy to show explicitly that

$$\mu \circ \omega = j: U \rightarrow (0) \times J^\oplus \rightarrow Q_1 S^0 \rightarrow SG.$$

Also $Ad^0_q$ has a transfer for cyclic coverings which admits an explicit bundle-theoretic description from which it is simple to see that $\mu$ commutes with cyclic covering transfers [3] of the two infinite loopspaces $J^\oplus$ and $SG$. Since $\delta$ extends to an exponential $H$-map

$$\delta: \bigcup_{n > 0} (n) \times J^\oplus \rightarrow \bigcup_{n > 0} Q_n S^0$$

which commutes with transfers for finite coverings, Theorem 3 is a consequence of the following result.

Theorem 5. There is a unique exponential $H$-map $\mu: \bigcup_{n > 0}(n) \times J^\oplus \rightarrow \bigcup_{n > 0} Q_n S^0$ which commutes with $p$-fold cyclic covering transfers and maps $(n) \times J^\oplus \rightarrow Q_n S^0$. 


Since \( \tau \) induces a unique \( \mu \) we may also deduce Theorem 3 from Theorem 1, once we acknowledge the existence of \( \tau \).

**Proof of Theorem 1 (cf. [7, II]).** \( \tau \) is described explicitly in terms of the geometry of fibre bundles of the form \( U(n)/N \to BN \to BU(n) \). The transfer on \( SG/U \) may be extended to the space \( \bigcup_{n \geq 0} (n) \times (SG/U) \). Furthermore \( \tau \) may be extended to an \( H \)-map

\[
\overline{\tau} : \bigcup_{n \geq 0} (n) \times BU(\Theta) \to \bigcup_{n \geq 0} (n) \times (SG/U),
\]

which maps \( (n) \times BU(\Theta) \) to \( (n) \times (SG/U) \). Also \( \overline{\tau} \) commutes with \( p \)-fold cyclic covering transfers. The proof of Theorem 1 is completed by means of the analogue of Theorem 5 for \( H \)-maps \( \bigcup_{n \geq 0} (n) \times J(\Theta) \to \bigcup_{n \geq 0} (n) \times (SG/U) \).

**References**


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