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*Methods of numerical integration*, by Philip J. Davis and Philip Rabinowitz, Academic Press, New York, 1975, 459 + xii pp., $34.50.

The study of numerical integration dates from antiquity right up to the present. It is an important topic in numerical analysis and scientific computing to which many mathematicians, scientists, and engineers have contributed.

Why numerical integration? Well, many integrals that arise in the real world simply cannot be evaluated analytically. And, of those integrals which can be analytically evaluated, the analytic "answer" may not be useful for computing. (An example: p. 2 of Davis and Rabinowitz.)

For functions of one variable, numerical integration is called "quadrature", from the Greek quadratos, meaning the square whose area equals the area under a given (positive) curve. For functions of more than one variable, numerical integration is called "cubature". Much more is known about quadratures, whereas cubatures are considerably more important to users, a standard state of affairs in mathematical subjects.

Numerical integration derives some of its appeal from the different levels of abstraction from which it can be approached. For example, functional analysis has been used to obtain error bounds. From classical real analysis, the beautiful theory of orthogonal polynomials leads to the powerful Gauss quadratures. On another hand, computing "rules of thumb" lead to the recent adaptive quadratures.

This reviewer feels that most functions of one variable can be adequately numerically integrated, interpolated, etc., but that many functions of more than one variable cannot, especially if numerical data instead of functions are involved. In the latter case, only data in very special geometric configurations can be handled, e.g., tensor or cross product data. Randomly placed data are treated by a Monte Carlo method, if at all.

On the positive side, what ideas from quadratures have been adapted to cubatures?

1. "Product" regions, such as cubes, cones, and cylinders: Cubatures for them can be built up from quadratures.

2. "Gauss cubatures": For the problem

$$\int_a^b f(x)\,dx \simeq \sum_{k=1}^{n} A_k f(x_k),$$

if the $2n$ unknowns consisting of the $A_k$ and $x_k$ are so chosen that the formula is exact whenever $f(x)$ is a polynomial of degree less than or equal to $2n - 1$, the integration scheme is called a Gauss quadrature. That is, a Gauss quadrature integrates exactly as many of the first monomials as there are parameters to be determined. The nodes (the $x_k$) of Gauss quadratures are the zeros of the corresponding orthogonal polynomials. Now, for functions of two variables and the problem

$$\iint_R f(x,y)\,dx\,dy \simeq \sum_{k=1}^{n} A_k f(x_k, y_k),$$

the nodes should be the common zeros of several orthogonal polynomials. The premier result is the special case: Radon's fifth degree formula with seven nodes (21 monomials and 21 parameters).

3. Remainder theory: The remainder is the (truncation) error. There are two main types: real or complex variable.

Quadrature remainders of the real variable type are usually of the form

$$R(f) = \int_a^b f^{(m)}(t) K(t)\, dt.$$

These are derived from Peano's Theorem (see p. 218 of Davis and Rabinowitz). $K(t)$ is called the Peano kernel and the parameter $m$ is less than or equal to the dimension of the quadrature's precision set, e.g., $m \leqslant 2n$ for Gauss quadratures. An issue only recently understood: People used to think that $m$ must equal the above dimension. This led to such claims as that Gauss quadratures should only be used for functions that were sufficiently smooth, namely $m = 2n$ and $f \in C^{2n}[a, b]$. A. H. Stroud and others have laid this bogie to rest. What *is* true is that $m = 2n$ sometimes leads to an especially simple form ("simplex" form) of the remainder, such as

$$R(f) = \alpha f^{(2n)}(\xi)/(2n)!, \qquad a < \xi < b,$$

for Gauss quadratures. Similarly for equally spaced quadratures, which are called Newton-Cotes rules.

Cubature remainder theory is covered by the elegant Sard kernel theory, a generalization of Peano's kernel theorem. The beautifully written source is Arthur Sard's *Linear approximation*, Math. Surveys, no. 9, American Mathematical Society, 1963. For an example of Sard's theory applied to cubatures, see the paper by Barnhill and Pilcher, *Sard kernels for bivariate cubatures*, Comm. ACM 16 (1973), 567–570.

The complex variable remainder theory involves a Hilbert space of analytic functions to which the integrand belongs. Then the majorized remainder is bounded in the "derivative-free" form $|R(f)| \leqslant \sigma \|f\|$ where $\sigma$ depends on $R$ but not on $f$.

We emphasize that both the real and complex variable error bounds can be *computed.* Bounds are always conservative, because they cover many functions of which $f$ is only one, but at least they are tabulated for many integration rules.

The numerical integration remainder theory applies equally well to other bounded linear functionals such as interpolation.

The authoritative book on cubature is A. H. Stroud's *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, N.J., 1971.

Davis and Rabinowitz is a monograph (not a textbook, e.g., no exercises), written to be "accessible to those with a background only in calculus". Its uniqueness lies in its footnotes and historical asides, although it covers quadrature well. The book has a large and useful bibliography, the size of which illustrates the activity in this field. The book also has a pleasant style, which enhances its readability.

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