FORM OF SOLUTIONS
TO THE $p$-ADIC EQUATION $y' = 0$

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The solutions of $y' = 0$ over the real and/or complex numbers have long been known to be the constant functions. Here we shall present a form for any function, $z$, that maps a suitable subset of $\mathbb{Q}_p$, the complete field of $p$-adic numbers, into $\mathbb{Q}_p$ (where $p$ is a positive prime) and is differentiable with derivative zero everywhere. We shall also discuss the image-set of such a function.

This problem has been of interest ever since J. Dieudonné gave an example [1, p. 90], [2, p. 19], [4, pp. 35, 39] of a function, $z_0$, that maps $\mathbb{Z}_p$ homeomorphically onto its image-set and yet has a zero derivative everywhere. M. van der Put has studied integration of $p$-adic valued functions [3] using the set of solutions to $y' = 0$ without determining this set [2, p. 20].

Let $N$ be the set of nonnegative integers, let $R$ be the set of real numbers, and let $R_{>b}$ be the set of all real numbers greater than the real number $b$. Let $C = \{0, 1, 2, 3, \ldots, p - 1\}$ and let $\mathbb{Z}_p$ be the set of $p$-adic integers. Every $p$-adic integer has a canonical form $\sum \binom{a_j p^j}{j \in N}$, where each $a_j$ is an element of $C$. $z_0$, the function of Dieudonné, is given by

$$z_0(\sum \binom{a_j p^j}{j \in N}) = \sum \binom{a_j p^{2j}}{j \in N}.$$ 

First we let $f$ be a function mapping a subspace of $\mathbb{Z}_p$ into $\mathbb{Z}_p$. It is easily shown that $f$ is uniformly continuous (on its domain) iff

$$(\exists l : N \rightarrow N)(\forall n \in N)(\exists g_n : C^{l(n)} \rightarrow \mathbb{Z}_p)(\forall a \in C^N)$$

$$\sum \binom{a_j p^j}{j \in N} \in \text{Dom}(f) \Rightarrow$$

$$f(\sum \binom{a_j p^j}{j \in N}) = \sum \binom{g_n(a_0, a_1, a_2, a_3, \ldots, a_l(n) - 1)p^n}{n \in N}.$$ 

Now the concept of uniform differentiability is introduced; it bears the same relationship to differentiability that uniform continuity has to continuity. Formally, $f$ is uniformly differentiable on $D'$ iff $D'$ is a subset of $\text{Dom}(f)$ that contains no isolated points of $\text{Dom}(f)$ and
558

(∀ ∈ ℝ>0) (∃ δ ∈ ℝ>0) (∀ x ∈ D') (∃ w ∈ Q_p) (∀ y ∈ Dom(f))

0 < |y - x| < δ => |(f(y) - f(x))/(y - x) - w| < ε.

Second, we show that a function f is uniformly differentiable with derivative zero everywhere if and only if it has the form (stated above) for uniformly continuous functions and \( n - l(n) \to \infty \) as \( n \to \infty \) (\( l \) may have to be chosen so that \( l(n) \) is the least value "that works"). Furthermore, the image-set of such a function has Jordan content zero, consequently it is nowhere dense.

However, there exist functions that map \( Z_p \) into \( Z_p \) and are differentiable with derivative zero everywhere, yet are not uniformly differentiable. One such function can be exhibited by putting \( C' = C\backslash\{0\} \) and defining \( z_1(x) \) on \( Z_p \) by \( z_1(x) = p^{2k} \) if, for some \( k \in \mathbb{N}, x \in C'p^k + C'p^{4k+4} + Z_p \) and \( z_1(x) = 0 \) otherwise. The least function \( l \) "that works" for \( z_1 \) is

\[
l(n) = \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
2n + 4 & \text{if } n \text{ is even.}
\end{cases}
\]

Third, a differentiable function with a continuous derivative on a measurable (with respect to a measure that is nonnegative, regular, and finite on compact sets) domain can have this domain expressed, except for a set of measure 0, as the union of a nondecreasing sequence of compact sets, on each of which the given function is uniformly differentiable (Shades of Egoroff's Theorem). As something of a converse, if \( f \) is a function and \( D \) is a nondecreasing sequence of sets on each of which \( f \) is uniformly differentiable, then \( f \) is differentiable on the union of \( D \).

Fourth and finally, the second and third results are combined to give a form for the solutions of \( y' = 0 \) over the \( p \)-adics.

In the course of the paper [5], uniform differentiability is related to other concepts of mathematics of the same ilk, e.g., uniform differentiability and bounded derivative imply uniform continuity.

Details will appear in the paper [5], and generalizations in another paper.

Both [5] and the current paper overlap slightly with [6, pp. 90–91]. There another form is given for a uniformly differentiable function defined on \( Z_p \) with derivative zero everywhere that takes its values in a valued field containing \( Q_p \).

REFERENCES


5. Gerald S. Stoller, *Form of solutions to the equation $y' = 0$ over $p$-adics* (to appear).


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