THE DIMENSION OF ALMOST SPHERICAL SECTIONS OF CONVEX BODIES

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The well-known theorem of Dvoretzky [1] states that convex bodies of high dimension have low dimensional sections which are almost spherical. More precisely, the theorem states that for every integer k and every $\epsilon > 0$ there is an integer $n(k, \epsilon)$ such that any Banach space X with dimension $\geqslant n(k, \epsilon)$ has a subspace Y of dimension k with $d(Y, l_k^2) \leqslant 1 + \epsilon$. Here $d(Y, l_k^2)$ denotes the Banach Mazur distance coefficient between Y and the k dimensional Hilbert space l_k^2 i.e. $\inf \|T\| \, \|T^{-1}\|$ taken over all operators T from Y onto l_k^2.

The estimate for $n(k, \epsilon)$ given in [1] was improved in [5] to $n(k, \epsilon) = e^{c(\epsilon)k}$.

In other words (considering the dependence of $n(k, \epsilon)$ on k for fixed ϵ) the dimension of the almost spherical section (of the unit ball) given by Dvoretzky's theorem is about the log of the dimension of the space. This estimate is in general the best possible, since as observed in [10] it is easy to verify that if $X = l_n^\infty$ any subspace Y of X whose Banach Mazur distance from a Hilbert space is $\leqslant 2$ say, must be of dimension at most $C \log n$. It turns out however that if we take into account not only X but also X^* much better information can be obtained.

THEOREM 1. *There is an absolute constant $c > 0$ so that for every Banach space X of dimension n the following formula holds*

(1)
$$k(X)k(X^*) \geqslant cn^2 \|P\|^2 / d^2(X, l_n^2).$$

This formula needs explanation. The number $k(X)$ is an integer with the following property: "most" subspaces Y of X of dimension $k(X)$ satisfy $d(Y, l_{k(X)}^2) \leqslant 2$. We do not intend to make the word "most" precise here. What really matters is that there exists a $Y \subset X$ with $d(Y, l_{k(X)}^2) \leqslant 2$. The number $\|P\|$ is the norm of a projection P. If $k(X) \leqslant k(X^*)$, P is a projection from X onto a suitable subspace Y of X with $d(Y, l_{k(X)}^2) \leqslant 2$ (again actually "most" subspaces Y of dimension $k(X)$ work). If $k(X^*) \leqslant k(X)$, P is a projection from X^* onto a subspace $Z \subset X^*$ with $d(Z, l_{k(X^*)}^2) \leqslant 2$.

The proof of Theorem 1 uses the same ideas as those employed by the third named author in [5]. The main tool used in the proof is the isoperimetric inequality for sets on the surface of the unit ball in n space (cf. [6]).

Since clearly $\|P\| \geqslant 1$ and $d(X, l_n^2) \leqslant \sqrt{n}$ (cf. [2]), Theorem 1 gives in particular that

(2) $k(X)k(X^*) \geqslant cn.$

(3) $\max(k(X), k(X^*)) \geqslant \sqrt{cn}.$

An easy consequence of (2) is the following result.

THEOREM 2. *There is an absolute constant $c > 0$ so that the following holds*: *Let Q be a convex polytope in R^n which is symmetric with respect to the origin and has the origin as an interior point. Let m be the number of vertices of Q and s the number of $(n-1)$-dimensional faces of Q. Then*

(4) $\log m \log s \geqslant cn.$

PROOF. Let X be the Banach space whose unit ball is Q. Then $X \subset l_s^\infty$ and $X^* \subset l_m^\infty$. By the remark made in the introduction concerning l_n^∞ spaces we have $k(X) \leqslant a \log s$, $k(X^*) \leqslant a \log m$ for some absolute constant a. Hence (4) follows from (2).

Easy examples show that the estimate in Theorem 2 is asymptotically the best possible. It is possible e.g. to construct convex symmetric polytopes in R^n (n arbitrarily large) so that $s \leqslant e^{a\sqrt{n}}$, $m \leqslant e^{a\sqrt{n}}$ for some constant a. Thus also (3) cannot be improved. The simplest examples which show that (2), (3) or (4) are the best possible are obtained by considering Banach spaces which are obtained from R (the 1-dimensional space) by repeated applications of finite direct sums in the l_∞ and l_1 norms.

It should be pointed out that Theorem 2 is actually true in stronger versions. For example, since we used (2), we ignored the terms $\|P\|$ and $d(X, l_n^2)$. Thus if e. g. the polytope Q is such that its distance d from an ellipsoid is better than the largest possible value $\sqrt{n}$ then we can improve (4) by replacing n by n^2/d^2.

The following simple example illustrates the use of $\|P\|$ in (1). Let X be the space $(l_n^\infty \oplus l_n^\infty \oplus \ldots \oplus l_n^\infty)_1$ (n terms in the direct sum which is taken in the l_1 sense). Then $\dim X = n^2$ and it is easy to verify by counting extreme points that $k(X^*) \leqslant cn$ and $k(X) \leqslant cn \log n$. It follows from (1) that X^* has a cn-dimensional subspace whose distance from Hilbert space is $\leqslant 2$ and on which there is a projection of norm $\leqslant \sqrt{\log n}$. This fact has an interesting application in terms of p-absolutely summing operators. There is a bounded linear operator from $(c_0 \oplus c_0 \oplus \ldots)_1$ into l_2 which is not 2-summing. This is surprising since every operator from c_0 into l_2 is 2-summing and every operator from l_1 into l_2 is even 1-summing.

By combining the proof of Theorem 1 with the Dvoretzky Rogers theorem we can obtain more information. The results become particularly interesting if we use the notions of type and cotype of a Banach space (cf. [4]). Here we state one result involving cotype. A Banach space X is said to be of cotype p ($p \geqslant 2$) if there is a constant K (called the cotype constant) so that for all choices of $\{x_i\}_{i=1}^m \subset X$ the average of $\|\Sigma_{i=1}^m \epsilon_i x_i\|$ taken over all choices of signs ϵ_i is $\geqslant$ $K(\Sigma_{i=1}^m \|x_i\|^p)^{1/p}$. Every uniformly convex space is of cotype p for some $p < \infty$.

THEOREM 3. *Let X be a Banach space of cotype p. Then there is a constant c (depending only on the cotype constant of X) so that for every $Y \subset X$ with $\dim Y = n$ we have $k(Y) \geqslant cn^{2/p}$ i. e. there is a $Z \subset Y$ with $\dim Z = k \geqslant cn^{2/p}$ and $d(Z, l_k^2) \leqslant 2$.*

It follows from a result of Krivine, Maurey and Pisier (cf. [4]) that a weak converse of Theorem 3 is true. If X satisfies the conclusion of Theorem 3 then X is of cotype $p + \epsilon$ for every $\epsilon > 0$. Since $L_p(0, 1)$, $1 \leqslant p \leqslant 2$ and C_p, $1 \leqslant p$ $\leqslant 2$ (the space of operators T on l_2 for which $\|T\|^p = \text{trace } (TT^*)^{p/2 < \infty}$) are of cotype 2 it follows from Theorem 3 that if we take any n linearly independent elements in one of these spaces then their linear span has a subspace of dimension cn (c an absolute constant) whose distance from Hilbert space is $\leqslant 2$.

Finally we mention an application which can be obtained by using Theorem 1, the proof of Theorem 3 and the arguments in [3]. It is the local version of the solution to the complemented subspaces problem.

THEOREM 4. *There is a real-valued function $f(\lambda)$ defined for $\lambda \geqslant 1$ so that the following holds: If X is a finite-dimensional space so that onto every $Y \subset X$ there is a projection of norm $\leqslant \lambda$ then $d(X, l_n^2) \leqslant f(\lambda)$ where $n = \dim X$.*

Further applications and detailed proofs will be published elsewhere.

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