ELLIPTIC PSEUDO DIFFERENTIAL OPERATORS
DEGENERATE ON A SYMPLECTIC SUBMANIFOLD

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1. Introduction. This note is concerned with the classes of pseudo
differential operators $L_{m,M}(\Omega, \Sigma)$, $\Sigma$ symplectic
submanifold of codimension 2, in Sjöstrand [4]; the definitions of $P$ in $L_{m,M}(\Omega, \Sigma)$ and of the associated wind-
ing number $N$ are recalled in §2. In Helffer [2] the study of the hypoellipticity
of $P$ is reduced to the analysis of the bounded solutions of an ordinary differential equation. Here we deduce an explicit result for $N = 2 - M$: essentially, we
can prove that in this case all the bounded solutions are products of an exponen-
tial function with polynomials.

2. The classes $L_{m,M}(\Omega, \Sigma)$ and the winding number. Let $\Omega \subset \mathbb{R}^n$ be an
open set. Let $\Sigma \subset T^*(\Omega)\setminus 0$ be a closed conic symplectic submanifold of co-
dimension 2 ($\Sigma$ symplectic means that the restriction of the symplectic form
$\omega = \Sigma d\xi_s \wedge dx_s$ to $\Sigma$ is nondegenerate). $L_{m,M}(\Omega, \Sigma)$ is the set of all the pseudo
differential operators $P$ which have a symbol of the form

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j/2}(x, \xi),$$

where $p_{m-j/2}$ is positively homogeneous of degree $m - j/2$ and for every $K \subset \subset \Omega$ there exists a constant $C_K$ such that

$$|p_m(x, \xi)|/|\xi|^m \geq C_K^{-1} d_\Sigma^M(x, \xi),$$

$$|p_{m-j/2}(x, \xi)|/|\xi|^{m-j/2} \leq C_K d_\Sigma^{M-j}(x, \xi), \quad 0 \leq j \leq M,$$

for all $(x, \xi) \in K \times \mathbb{R}^n$, $|\xi| > 1$ ($d_\Sigma(x, \xi)$ is the distance from $(x, \xi/|\xi|)$ to $\Sigma$).

Fix $\rho$ in $\Sigma$, denote by $N_\rho(\Sigma)$ the orthogonal space of $T_\rho(\Sigma)$ with respect
to $\omega$ and choose two linear coordinates on $N_\rho(\Sigma) u_1, u_2$ such that $\omega/N_\rho(\Sigma) =
du_2 \wedge du_1$. Take $X = (u_1, u_2) \in N_\rho(\Sigma)$ and let $V$ be any vector field on $T^*(\Omega)$
equal to $X$ at $\rho$. We define the homogeneous polynomial

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\[ c \prod_{h=1}^{M} (u_2 - r_h u_1) = \frac{1}{M!} (V^M p_m)_p. \]

In view of (2) \( \text{Im} \ r_h \neq 0 \) for each \( h, h = 1, \ldots, M \): let \( M^+ (M^-) \) be the number of the \( r_h \)'s such that \( \text{Im} \ r_h > 0 (\text{Im} \ r_h < 0) \). The integer \( N = M^+ - M^- \) may take the values \( M, M - 2, \ldots, 2 - M, -M \) ; here we assume \( M \geq 2 \) and

\[ N = 2 - M, \text{ for every } \rho \text{ in } \Sigma. \]

3. The problem of the hypoellipticity. Let \( P \in L^{m,M}(\Omega, \Sigma) \) satisfy (5).

We are interested in the following hypoellipticity property:

\[ \text{For any open subset } U \text{ of } \Omega \text{ and any distribution } f \text{ in } U, Pf \in H^s_{\text{loc}}(U) \text{ implies } f \in H^{s+m-M/2}_{\text{loc}}(U). \]

Let \( \mathcal{P} \) be the algebraic vector space of all the polynomials in one real variable with complex coefficients and denote by \( L(\mathcal{P}) \) the space of all the linear maps from \( \mathcal{P} \) into \( \mathcal{P} \). We associate to \( \mathcal{P} \) an application \( A_p : \rho \in \Sigma \rightarrow A_p(\rho) \in L(\mathcal{P}). \)

The explicit definition of \( A_p \) will be given in \$4\; \text{; first let us state our main result.}

**Theorem 1.** Let \( P \in L^{m,M}(\Omega, \Sigma) \) satisfy (5). Then (6) holds if and only if

\[ \text{dimension Ker } A_p(\rho) = 0, \text{ for every } \rho \text{ in } \Sigma. \]

4. Definition of \( A_p(\rho) \). If (5) is satisfied, then it is \( M^+ = 1 \) and \( M^- = M - 1 \): we will assume \( \text{Im} \ r_h < 0 \) for \( 2 \leq h \leq M \) and \( \text{Im} \ r_1 > 0 \). As in Helffer [2], initially we construct a family of ordinary differential operators with polynomial coefficients. Consider the symbol \( q(x, \xi) \) with asymptotic expansion

\[ \sum_{j=0}^{\infty} q_{m-j/2} \sim \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left( \sum \frac{\partial^2}{\partial x_\alpha \partial \xi_\beta} \right)^i p. \]

Using the notations of \$2\, \text{, we define on } N_{\rho}(\Sigma) \text{ the polynomial (the leading part coincides with (4))}

\[ c \prod_{h=1}^{M} (u_2 - r_h u_1) + \sum_{\alpha + \beta < M} c_{\alpha, \beta} u_1^\alpha u_2^\beta = \sum_{j=0}^{M} \frac{1}{(M-j)!} (V^{M-j} q_{m-j/2})_p. \]

We rewrite the left-hand side of (8) in the symmetric form

\[ \sum_{\gamma(h), \delta(h) \leq M} c_{\gamma(h)}' u^{\gamma(h)}, \]

where the components of the multiorder \( \gamma(h) = (\gamma_1, \ldots, \gamma_h) \) may take the value 1 or 2, \( c_{\gamma(h)}' = c_{\delta(h)}' \) if \( |\gamma(h)| = |\delta(h)| \) and we have noted

\[ u^{\gamma(h)} = u_{\gamma_1} u_{\gamma_2} \cdots u_{\gamma_h}. \]
Now, maintaining the order of the factors in (10), we replace $u_2$ in (9) by $D = -id/du_1$. We get a differential operator $M(\rho)$ which can be expressed in the form

$$M(\rho) = c(D - r_Mu_1) \cdots (D - r_1u_1) + \sum_{\alpha + \beta < M} c''_{\alpha, \beta} u_1^\alpha D^\beta.$$  

Set

$$\eta = -i \sum_{\alpha + \beta = M-1}^{c''_{\alpha, \beta} r_1^\beta / c} \left( \prod_{h=2}^M (r_1 - r_h) \right).$$

We define for $Q \in \mathcal{P}$,

$$A_\rho(Q(u_1)) = \exp(-ir_1u_1^2/2 - \eta u_1)$$  

$$= \exp[c(\rho) + \eta u_1] Q(u_1)$$

The definition of $A_\rho(\rho)$ depends on the initial choice of the coordinates $u_1, u_2$. We can prove that, starting from other canonical coordinates $u'_1, u'_2$ and repeating the construction, we get a map $A'_\rho(\rho)$ such that $U^{-1}(\rho)A'_\rho(\rho)U(\rho) = A_\rho(\rho)$ for some automorphism $U(\rho)$ in $L(\mathcal{P})$. Therefore condition (7) has an invariant meaning.

5. Applications. Take $Q(u_1) = \sum_{\nu=0}^k c_{\nu} u_1^\nu$. Developing (13) we obtain

$$A_\rho(Q(u_1)) = \sum_{\mu=0}^{k+M-2} \left( \sum_{\nu=0}^k d_{\mu, \nu} b_\nu \right) u_1^\mu,$$

where $d_{\mu, \nu}$ are polynomials in the variables $c, r_1, \ldots, r_M, c''_{\alpha, \beta}$. We write $D^{(k)}$ for the matrix $(d_{\mu, \nu})$, $\mu = 0, \ldots, k + M - 2$, $\nu = 0, \ldots, k$. Let $\sigma = \{\mu_0, \mu_1, \ldots, \mu_k\}$ be a subset of $\{0, 1, \ldots, k + M - 2\}$ and let $D_\sigma^{(k)}$ denote the minor $(d_{\mu, \nu}^t)$, $t = 0, \ldots, k, \nu = 0, \ldots, k$. Theorem 1 can be rewritten in the following way.

**Theorem 2.** Let $P \in L^{m,M}(\Omega, \Sigma)$ satisfy (5). Then (6) holds if and only if for each fixed $\rho \in \Sigma$ and for every integer $k \geq 0$ there exists a subset $\sigma = \{\mu_0, \mu_1, \ldots, \mu_k\}$ of $\{0, 1, \ldots, k + M - 2\}$, such that

$$\det D_\sigma^{(k)} \neq 0.$$  

A direct computation shows that for $\sigma_0 = \{M - 2, M - 1, \ldots, k + M - 2\}$

$$\det D_{\sigma_0}^{(k)} = \lambda(\rho) \prod_{\nu=0}^{k} [\ell(\rho) - \nu],$$

where $\lambda(\rho), \ell(\rho)$ are rational functions of $c, r_1, \ldots, r_M, c''_{\alpha, \beta}$: $\lambda(\rho) \neq 0$ and $\ell(\rho)$ coincides with the invariant in Boutet de Monvel and Treves [1] and Helffer [3].
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